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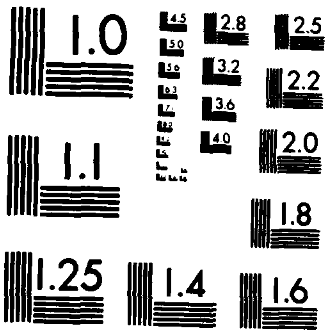

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Shock-Free Transonic Airfoil Design by a Hodograph Method

William Zeitler Strang, 2Lt, USAF. M.S. Thesis, University of Vermont, 85 pp., 1984.

ABSTRACT

Refined mathematical methods are required for the analytical solution of the partial differential equation governing steady, two-dimensional, compressible, transonic, potential fluid flow. This equation is nonlinear in the physical plane and so does not lend itself to standard analytical solution methods. The Molenbroek-Chaplygin transformation, where the physical Cartesian coordinates as the independent variables are replaced by the velocity magnitude and direction as the independent variables, linearizes the governing equation which may then be analytically solved. The plane where the said velocity parameters are the independent variables is termed the hodograph plane. Likewise, the transformed differential equation is known as the hodograph equation and it is solved by hodograph methods.

This mathematical study addresses the solution of transonic flow phenomena by an extension of Lighthill's hodograph method. Lighthill's method transforms a given solution of the Laplace equation into a solution of the hodograph equation for subsonic flows only. A new relation is developed in this study extending this transformation technique to include flows up to Mach 2.2735 in air. Requiring only numerical data concerning the velocity field, this hodograph method is computationally efficient and mathematically straightforward.

Primary Sources

Lighthill, M. J., The Hodograph Transformation in Trans-sonic Flow. I. Symmetrical Channels. II. Auxiliary Theorems on the Hypergeometric Functions. III. Flow Round a Body. Proc. Royal Soc., London. Vol A191 (1947), pp. 323-369.

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SHOCK-FREE TRANSONIC AIRFOIL

DESIGN BY A HODOGRAPH METHOD

A Thesis Presented

by

William Zeitler Strang

to

The Faculty of the Graduate College

of

The University of Vermont

In Partial Fulfillment of the Requirements
for the Degree of
Master of Science

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Accepted by the Faculty of the Graduate College, the University of Vermont, in partial fulfillment of the requirements for the degree of Master of Science, specializing in Mechanical Engineering.

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Date: April 20, 1984

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List of Symbols Used

x, y	cartesian coordinates in the physical plane
u, v	velocity components in the x and y directions, respectively
q, e	velocity magnitude and angle, respectively
c	local sonic velocity
Cp	specific heat at constant pressure
γ	ratio specific heats
T	temperature
R	ideal gas constant
p	pressure
ρ	density
M	Mach number
Γ	circulation strength
B	doublet strength
Z	complex physical coordinate
ζ	complex velocity
ζ^*	complex velocity at the branch point of the hodograph plane
ψ	stream function

ϕ	potential function
Φ	complex potential function
ψ_n	Chaplygin functions
r_m, P_m	the m^{th} residue and the m^{th} pole, respectively
$f(n,\tau_\infty)$	normalization function
τ	ratio of local speed to maximum attainable speed
s	subsonic speed parameter
β, t	supersonic speed parameters
σ	value of subsonic speed parameter at the sonic speed

Subscripts and Superscripts

i denotes the incompressible case.

c denotes the case of flow about a circle.

s denotes sonic conditions.

o denotes stagnation conditions.

oo denotes free stream conditions.

A variable as a subscript indicates the partial derivative with respect to that variable.

Chapter 1

INTRODUCTION

Problem Overview

The differential equation in terms of the potential function, ϕ , governing the steady, two-dimensional, inviscid, irrotational, isentropic flow of an ideal gas is

$$[1-\frac{\phi_x^2}{c^2}]\,\phi_{xx}+[1-\frac{\phi_y^2}{c^2}]\,\phi_{yy}-2\,\frac{\phi_x\phi_y}{c^2}\,\phi_{xy}=0 \tag{1.1}$$

where the local speed of sound, c , is

$$c^2=c_o^2-(\gamma-1)/2\;(\phi_x^2+\phi_y^2)\ . \tag{1.2}$$

In terms of the stream function, ψ , (1.1) and (1.2) are

$$[1-(\frac{\rho_o}{\rho})^2\frac{\psi_y^2}{c^2}]\psi_{xx}+[1-(\frac{\rho_o}{\rho})^2\frac{\psi_x^2}{c^2}]\psi_{yy}+2(\frac{\rho_o}{\rho})^2\frac{\psi_x\psi_y}{c^2}\psi_{xy}=0 \tag{1.3}$$

$$c^2=c_o^2-\frac{\gamma-1}{2}(\frac{\rho_o}{\rho})^2(\psi_x^2+\psi_y^2)\ . \tag{1.4}$$

For a derivation of the above results, consult Appendix A and/or Shapiro[1]. The non-linear behavior of differential equations (1.1) and (1.3) is evident.

In the limit of vanishing compressibility, $c^2\longrightarrow\infty$ and equations (1.1) and (1.3) reduce to the Laplace equations

$$\phi_{xx} + \phi_{yy} = 0$$

$$\psi_{xx} + \psi_{yy} = 0 .$$

The potential transform of Legendre and the Molenbroek-Chaplygin transformataion linearize equations (1.1) and (1.3). This study addresses the solution of the partial differential equation resulting from the latter transformation.

Considering the physical Cartesian coordinates x and y as functions of the potential and stream functions, one may deduce the hodograph equations

$$\phi_\theta = q(\rho_o/\rho) ; \quad \phi_q = q(\rho_o/\rho q)\psi_\theta . \tag{1.5}$$

Defining,

$$\tau \equiv (q/q_m)^2$$

where q_m is the maximum velocity attainable when all the flow's internal energy is converted to kinetic energy, the governing hodograph equation is

$$PQ\psi_{\tau\tau}+PQ_\tau\psi_\tau-\psi_{\theta\theta} = 0 \tag{1.6}$$

$$PQ = 4\tau^2(1-\tau)/(1-\tau/\tau_s) \tag{1.7}$$

$$PQ = -4\tau[1+(2-\gamma)/(\gamma-1)]/(1-\tau/\tau_s) \tag{1.8}$$

$$\tau_s \equiv (\gamma-1)/(\gamma+1) = \text{Mach one.} \tag{1.9}$$

The above equations are derived in Appendix B.

The solution of equation (1.6) presents two distinct difficulties. First, (1.6) is a partial differential of the mixed type. Consider its discriminant:

$$B^2 - 4AC = 0 -4(PQ)(-1) = 4PQ .$$

In subsonic flow, $0<\tau<\tau_s$, the discriminant is negative and hence equation (1.6) is elliptic. In supersonic flow, $\tau_s<\tau<1$, the discriminant is positive and (1.6) is now hyperbolic. This change in behavior elaborates the solution method considerably. Second, the Jacobian of the Molenbroek-Chaplygin transformation may be zero or infinite at specific points or along lines in the hodograph plane. The transformation is, therefore, no longer one-to-one. See Appendix B and/or Ferrari and Tricomi[2] and Lighthill[3] for further remarks concerning the Jacobian of the transformation.

Chaplygin, in 1904, solved equation (1.6). The solution, a function of τ, θ, and arbitrary complex separation constant, n, is

$$\psi_n(\tau,\theta)=\tau^{n/2}F(a_n,b_n;n+1;\tau)e^{\pm in\theta} \quad (1.10)$$

The hypergeometric function is denoted as $F(a_n,b_n;n+1;\tau)$ and the product $\tau^{n/2}F(a_n,b_n;n+1;\tau)$ is termed the Chaplygin function. The reader is referred to Appendix B for the derivations and details of (1.10). Bergman[4] noted that any particular solution (1.10) corresponding to a particular n will generally converge in only part of the hodograph plane and that the complete solution for the entire hodograph plane is

$$\psi(\tau,\theta)=\mathrm{Im}\{\sum_{n=o}^{\infty}\psi_n(\tau_o\theta)\} \quad (1.11)$$

Appendix C and/or Boerstoel[5] should be consulted for details of the hodograph plane.

In light of Bergman's result and of the fact that all boundary conditions are lost under the Molenbroek-Chaplygin transformation, Lighthill[3] posed

$$\psi(\tau\theta)=\mathrm{Im}\{\sum_{n=o}^{\infty}\psi_n(\tau)f(n,\tau_\infty)e^{-in\theta} \tag{1.12}$$

as a solution to equation (1.6). The normalizing function, $f(n,\tau_\infty)$, is chosen such that (1.12) tends towards a solution of the Laplace equation in the limit of vanishing compressibility. This Laplace solution, in effect, represents the boundary conditions. Lighthill developed a method where a solution to the Laplace equation, given in terms of the hodograph variables q and θ, is transformed into a solution of the hodograph equation (1.6). For subsonic flow, the transformation is especially elegant, requiring only numerical data concerning the incompressible velocity field. Supersonic flow regions require a Laurent series expansion representation of the Laplace solutions in terms of q and θ. Nieuwland[6] generalized the representation to include Mellin-Barnes integrals. Both representations require highly advanced mathematics to properly represent only the simplest of Laplace solutions in the hodograph plane. In fact, the Laplace solution governing the incompressible flow about a general lifting airfoil cannot be represented by either method. The trailing-edge closure problems experienced by all who employ the hodograph method are symptomatic of this fact.

The key to this dichotomous behavior of the transformation lies in Lighthill's asymptotic forms of the Chaplygin functions as $|n|\longrightarrow\infty$. Because equation (1.6) is elliptic for $0<\tau<\tau_s$ and hyperbolic for $\tau_s<\tau<1$, different asymptotic formulae are required in each region. While the assumptions behind Lighthill's subsonic asymptotic formula are physically sensible, those behind his supersonic asymptotic formulae are not.

Objective

A physically reasonable asymptotic form of the Chaplygin functions as $|n| \longrightarrow \infty$ and valid for $(\gamma-1)/(\gamma+1)<\tau<.5083$ in air is developed. A transformation, entirely analogous to Lighthill's is derived where only numerical data concerning the incompressible velocity field are required. This permits the transformation of far more complicated and physically realistic Laplace solutions (hereafter termed "model flows"), than was previously possible. In particular, incompressible flows about closed lifting profiles can be transformed to represent transonic compressible flows about affinely related profiles. Lastly, a numerical method is developed which calculates the required data about any given profile.

Chapter 2

LITERATURE SURVEY

In 1890, the Dutch mathematician Molenbroek linearized the governing equations of motion by considering the potential and stream functions as functions of the velocity coordinates.

Chaplygin in 1904 derived the solution to the most studied version of the linearized partial differential equation. In this method the strema function is represented by an infinite series of particular solutions each of which converges in part of the domain of the flow. The series representing the strema function is the product of a hypergeometric series and a velocity magnitude parameter. Manipulation of such functions requires rather advanced mathematics. Chaplygin noted that by specifying

$$(\rho_o/\rho)\ \sqrt{M^2-1} = 1$$

the equation of motion reduces to the Laplace equation. This amounts to specifying a fictitious gas with $\gamma = -1$.

Meyer in 1908 found a "lost solution" which is the expansion of a flow around a corner.

Demtchenko and Busemann in 1932 and 1937, respectively, realized that Chaplygin's suggestion of $\gamma = -1$ amounts to replacing the curve of the ideal gas isentrope in the p vs. $1/\rho$ plane by a straight line. They limited themselves to considering this straight line as a tangent to the true isentrope at stagnation conditions. This restricted their work to flows of Mach number less than about .3.

Von Karman and Tsien extended the above method by taking the tangency point to coincide with free stream conditions. This shifted the range of usefulness to higher subsonic Mach numbers.

Tsien in 1939 appears to be the first to consider transforming solutions of Laplace's equation to solutions of the mixed equation. He derived a relation between a line element in the complex physical plane and the compressible complex potential. His last step is the replacement of the compressible complex potential by that corresponding to the incompressible case. Thus, a transformation from the incompressible physical plane to the compressible plane via the hodograph was achieved.

Ringleb in 1941 found another which is the flow about a sharp edge.

In 1946 and 1947, Tomotika and Tamada[7] developed an approximation to the ideal gas isentrope. Their "isentrope" coincides with the ideal gas isentrope precisely at the sonic point and to the order of their tangents at the stagnation point. The resulting partial differential equation for the stream function in the hodograph is easily solved. The solution is the product of Bessel functions and trigonometric functions. They calculated a shock-free transonic flow about a certain airfoil-like obstacle without circulation.

Nocilla in 1954-6 used the Tomotika-Tamada gas to calculate the shock-free transonic flow about airfoils with blunt noses. Circulation is again absent.

There are numerous examples of researchers approximating the true ideal gas isentrope by another curve rendering the solution easier to manipulate than the Chaplygin functions. One of them, the Tricomi equation replaces the ideal gas isentrope with a straight line and the regular solutions are Airy functions.

Bergman[4] in 1945 appears to be the first to successfully treat the exact case. The method is based on linear integral operations and is quite complex. Furthermore, circulation is still not included.

Cherry[8] in 1947 solved the exact equation resulting not from the Molenbroek-Chaplygin transformation but from the Legendre potential transformation. In this transformation correspondence between the physical plane and the hodograph ismore direct than in the Molenbroek-Chaplygin transformation, but the unknown quantities lack the physical basis of the stream and potential functions present in the Molenbroek-Chaplygin transformation. Cherry calculated the non-circulatory flow about a circular cylinder.

Lighthill[3] in 1947 is the first to solve the exact equation resulting from the Molenbroek-Chaplygin transformation. He first developed the asymptotic formulae for the Chaplygin functions as the magnitude of the complex separation constant tends towards infinity. For subsonic flow only, Lighthill built an entire function in the complex plane of the separation constant. This entire function is composed of two terms which turn out to be equal to one another by consideration of poles, residues, the maximum modulus theorem. Lastly, Lighthill developed the compressible stream function through a transformation of the incompressible complex potential.

That this method even exists is due to the form and simplicity of the subsonic asymptotic formula for the Chaplygin functions. Lighthill showed that if a general Laurent series for the incompressible flow exists in the hodograph and is convergent at the sonic speed, then the equation for the compressible stream function in subsonic flow may be analytically continued into the supersonic region of the hodograph plane. Such a Laurent series representing the incompressible complex potential about a practical airfoil in the hodograph plane will be very difficult if not impossible to derive and manipulate.

In Lighthill's transformation technique, a normalizing function is introduced that forces the solution of the mixed governing equation to reduce to solutions of Laplace's equation in the limit of vanishing compressibility. Lighthill's method can treat flows with circulation via the requirement that as one encircles the airfoil in the hodograph plane, all transformed variables remain single-valued. This limits the choice of the normalizing function to one particular example which Lighthill found.

Nieuwland[6] in 1967 showed that Mellin-Barnes integral representations of the incompressible complex potential are valid, and preferred, alternatives to the Laurent series representations. Nieuwland employed the Mellin-Barnes integrals to properly model considerably more complicated flows than could be handed by a Laurent series. He calculated the flow about a family of quasi-elliptical airfoils with circulation.

Bauer, Gorabedian, and Korn[9] in 1972 solved the transonic flow problem by rewriting the two basic governing equations in a complex form. These two equations are then decoupled into a system of linear ordinary differential equations which they solve by finite differences. Bauer et al. show they can compute a wide variety of advanced airfoils.

Boerstoel[5] in 1977 further extended the method of Nieuwland. He used the incompressible complex potential Nieuwland employed to generate a "basic stream function" which possessed the required singularities and basic properties of the incompressible complex potential about any general airfoil. To the basic stream function, Boerstoel added the "additional stream function" which also satisfied the governing equation and represented the complex incompressible potential about a non-lifting circular cylinder. An assumed arbitrary airfoil image in the hodograph plane permits solution of the additional stream function by a Tricomi boundary value problem treatment such that the sum of the basic and additional stream functions yields the pre-chosen airfoil image. Boerstoel's method produces realistic airfoils.

Takanaski in 1971 and Shigemi[10] in 1981 constructed airfoils similar to Nieuwland's using his technique. Shigemi introduced the "YC" profile to cope with the problem of trailing edge closure which plagued Nieuwland, Bauer et al., Takanaski, and Boerstoel. Shigemi correctly reasoned that the problem lies in the model flow, which is transformed. The "YC" profile does not solve the problem. It is, however, an extremely valuable advance. With relative ease, the transonic flow about a wide variety of cambered lifting airfoils is calculated. These airfoils are similar to Joukowski airfoils.

Chapter 3

ASYMPTOTIC FORMULA DEVELOPMENT

Lighthill's incompressible flow to subsonic compressible flow transformation depends upon two facts. First, any solution of the hodograph equation (1.6) must, in the limit of vanishing compressibility, reduce to a solution of the Laplace equation in hodograph variables. Second, the maximum value of a function analytic and not constant in a domain occurs on the boundary of that domain by the maximum modulus theorem.

Consider the first point. For strictly subsonic flow, the characteristic equation (see Appendix D) is

$$d\theta/\partial\tau=(-PQ)^{-1/2} \tag{3.1}$$

which when integrated yields:

$$s = \sigma + \sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma+1)(1-\tau)}} + \tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}} \tag{3.2}$$

where σ is an arbitrary constant. The hodograph equation for strictly subsonic flow is

$$\psi_{ss}+\psi_{\theta\theta}=T(s)\psi_s \tag{3.3}$$

where

$$T(s)=\frac{-1}{2}(-PQ)^{1/2}\left(\frac{Q_\tau}{Q}-\frac{P_\tau}{P}\right)=\frac{2(\gamma+1)}{(\gamma-1)^2}\tau^2(1-\tau)^{-1/2}\left(1-\frac{\gamma+1}{\gamma-1}\tau\right)^{3/2} . \tag{3.4}$$

Thus, s and T are both analytic functions of τ in the domain $\tau<\tau_s$. Since the derivative of s with respect to τ is not zero in the domain, τ is an analytic function of s by the inversion theorem. Since $\tau\neq\tau_s$ or 1 in said domain, T is also analytic function of s. For very small τ, the Chaplygin functions behave as $\tau^{n/2}$. Because the Chaplygin functions comprise at least part of the compressible flow solution, the functional form of τ with respect to s is needed so that the compressible solution can be forced to the Laplace solution in the limit of vanishing compressibility. Lighthill[3] chose the value of σ to be that value which causes "τ to be asymptotically e^{2s} as $\tau\longrightarrow 0$ and $s\longrightarrow -\infty$, σ is that value of s at $\tau = (\gamma-1)/(\gamma+1)$, the sonic speed; . . ." That value of σ is

$$\sigma= \frac{1}{2}\ln|2(\gamma-1)|-\sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}};\ \sigma= -1.173 \text{ for } \sigma=1.4. \quad (3.5)$$

Details of the derivation are in Appendix D.

Lemma 1 (Lighthill[3]). Both τ and T are analytic functions of e^{2s} in the region $|e^{2s}|<e^{2\sigma}$.

Proof . . . $e^{2s}=\tau$ . . . is an analytic function of τ in any domain excluding $\tau=(\gamma-1)/(\gamma+1)$ and $\tau=1$; its derivative is not zero in such a domain; and at neither of the singular points can $|e^{2s}|$ be less than $e^{2\sigma}$. Hence, by the inversion theorem, τ is an analytic function of e^{2s} in the region $|e^{2s}|<e^{2\sigma}$. As τ is never 1 or $(\gamma-1)/(\gamma+1)$ is the region, T must also be analytic.

For strictly supersonic flow, the above incompressible boundary condition does not exist. Integrating the characteristic equations of supersonic flow yields:

$$t_1 = \pm\{\varepsilon + \sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{(\gamma+1)\tau-(\gamma-1)}{(\gamma+1)(1-\tau)}} - \tan^{-1}\sqrt{\frac{(\gamma+1)\tau-(\gamma-1)}{(\gamma-1)(1-\tau)}} \tag{3.6}$$

while the hodograph equation for purely supersonic flow is

$$\psi_{tt}+S(t)\psi_t=\psi_{\theta\theta} \tag{3.7}$$

where

$$s(t)= \pm(PQ)^{1/2}\left(\frac{Q_\tau}{Q} - \frac{P_\tau}{P}\right)= \pm\frac{2(\gamma+1)}{(\gamma-1)^2}\,\tau^2(1-\tau)^{-1/2}\left(\frac{\gamma+1}{\gamma-1}\,\tau-1\right)^{-3/2} \tag{3.8}$$

The value of ε is that value such that τ is asymptotically $e^{\pm 2t_1}$ as $\tau \longrightarrow (\gamma-1)/(\gamma+1)$. At a glance

$$\varepsilon = 1/2\,\ell n\,|\gamma-1/\gamma+1| \;;\quad \varepsilon = -.8959 \text{ for } \gamma = 1.4 \,. \tag{3.9}$$

Appendix D contains further details concerning the supersonic flow equations where $t = t_1 - \varepsilon$.

Lemma 2. Both τ and S are analytic functions $e^{\pm 2t}$ in the domain $\tau_s < \tau < 1$.

Proof. By equation (3.6), $e^{\pm 2t_1} = \tau + \ldots$ is an analytic function of τ in the domain $\tau_s < \tau < 1$. No singular points exist in this domain and its derivative is not zero in the domain. By the inversion theorem τ is an analytic function of $e^{\pm 2t_1}$ in said domain. Because $e^{2\varepsilon}$ is a constant, then τ is also an analytic function of $e^{\pm 2t}$. Since S is an analytic function of τ , it is also an analytic function of $e^{\pm 2t}$.

Consider now the implications and requirements of the maximum modulus theorem. The Chaplygin functions are analytic with respect to n except at $n = -2, -3, -4 \ldots$, where they have simple poles.

Theorem 1 (Lighthill[3]). If $0<\tau\leq 1$, $\psi_n(\tau)$ is an analytic function of n except at $n = -2, -3, -4, \ldots$, where it has simple poles, its residue at $n = -m$ being $-mC_m\psi_m(\tau)$, where $C_m \ldots$ is positive and $\sim (2\pi m)^{-1}e^{-2\sigma m}$ as $m \longrightarrow \infty$.

The sign, $\sim$, means "asymptotes to." Assume two functions defined on the entire complex plane can be constructed which possess identical poles and residues at those poles. Their difference will be an entire function. The maximum value occurs at infinity by the maximum modulus theorem. If this maximum value is zero, by the maximum and minimum modulus theorems, the two functions are equal to each other on the entire complex plane. Lighthill employed this fact in his incompressible-to-compressible transformation where one of the two functions is constructed of Chaplygin functions. The forms of the Chaplygin functions as $|n| \longrightarrow \infty$, called asymptotic forms, are required to ensure the maximum modulus is zero. The asymptotic forms change as the flow changes from subsonic to supersonic.

Subsonic

Integrate $T(s)$ to produce $V(\tau)$:

$$V(\tau)=(-P/Q)^{1/4}=\mathrm{EXP}\{1/2\int_{\infty}^{s} T((s_1)ds_1\} \quad . \tag{3.10}$$

Thus,

$$V(0)=1;dV(\tau)/ds=TV/2 \ . \tag{3.11}$$

Assuming

$$\psi_n(\tau)=e^{ns}V(\tau)W_n(s) \qquad (3.12)$$

the hodograph equation for strictly subsonic flow (3.3) becomes

$$d^2W_n/dS^2+2ndW_n/ds=[1/4T^2(s)-1/2dT(s)/ds]W_n \ . \qquad (3.13)$$

Lighthill determined that as $|n|\longrightarrow\infty$ and excluding the negative integers, $W_n(s)\longrightarrow 1$.

Theorem 2 (Lighthill[3]). If $\delta>0$ and $\sigma_1<\sigma$, then $W_n(s) \quad 1$, i.e. $\psi_n(\tau)\sim e^{ns}V(\tau)$, uniformly for $s\leq\sigma_1$ and for n in the whole complex plane with circles of radius δ around each negative integer excluded, as $|n|\longrightarrow\infty$.

Consult Appendix D for the complete derivation of theorem 2.

Supersonic

For $\tau_s<\tau<1$, define:

$$\Omega(\tau)=(P/Q)^{1/4}+\text{ constant}=\text{EXP}\{-1/2\int_o^t S(t_2)dt_2\} \qquad (3.14)$$

where

$$\Omega(\tau_s)=1 \ ; \ d\Omega(\tau)/dt= -S\Omega/2 \ . \qquad (3.15)$$

Assume,

$$\psi_n(\tau)=e^{nt}\Omega(\tau)\ L_n(t)e^{n\varepsilon} \ . \qquad (3.16)$$

Since $\psi_n(\tau)$, e^{nt} , and $\Omega(\tau)$ are analytic functions of τ and e^{2t} by lemma 2, so too must $L_n(t)$. Thus,

$$L_n(t) = \sum_{r=o}^{\infty} l_{n,r}e^{2rt} \ . \qquad (3.17)$$

Note from equation (3.16) $\lim_{t\to o} L_n(t)e^{n\varepsilon} = \psi_n(\tau_s)$. Subjecting equation (3.16) to the linear operator of equation (3.7) yields:

$$d^2L_n(t)/dt^2 + 2ndL_n(t)/dt =$$

$$[1/4S^2(t)+1/2\ dS(t)/dt\ -2n^2]\ L_n(t) \qquad (3.18)$$

By lemma 2

$$1/4S^2(t)+1/2\ dS(t)/dt = \sum_{r=o}^{\infty} S_r e^{2rt}\ . \qquad (3.19)$$

Substituting equations (3.17) and (3.19) into (3.18) yields:

$$\sum_{r=1}^{\infty} 4r^2 \ell_{n,r} e^{2rt} + 2n \sum_{r=1}^{\infty} 2r\ell_{n,r} e^{2rt} = \sum_{r=o}^{\infty} s_r e^{2rt} - 2n^2 \sum_{r=o}^{\infty} \ell_{n,r} e^{2rt} \qquad (3.20)$$

Equating powers of e^{2t}

$$4r(n+r)\ell_{n,r} = \sum_{m=o}^{r} s_{r-m}\ell_{n,m} - 2n^2\ell_{n,r} \qquad (3.21)$$

$$= \sum_{m=o} S_{r-m}\ell_{n,m} + (s_o - 2n^2)\ell_{n,r}\ . \qquad (3.22)$$

Equating the e^o term yields $S_o = 2n^2$ or $\ell_{n,o} = 0$. Assuming $\ell_{n,o} = 0$ implies that any general $l_{n,r}$ is a function of $l_{n,1}$. Since $l_{n,1}$ is not restricted in any way, $S(t)$ is implied to be a function of an arbitrary constant which is false. Thus, $S_o = 2n^2$ and

$$4r(n+r)\ell_{n,r} = \sum_{m=o}^{r-1} S_{r-m}\ell_{n,m} \qquad (3.23)$$

This equation predicts the occurrence of poles at negative integral n for $\psi_n(\tau)$ as $\ell_{n,r}$ becomes indeterminate at these points. At first glance, the assumption of $S_o = 2n^2$ would seem to imply $S(t) = f(n)$ which is false. However, the implication that the coefficients S_r are functions of n does not guarantee that $S(t)$ is a function of n. Lighthill also must use this observation in his development of the asymptotic formula of the Chaplygin functions for subsonic flow. By $\lim_{t\to o} L_n(t)e^{n\epsilon} = \psi_n(\tau_s)$ equation (3.16) predicts

$$e^{n\varepsilon} \sum_{r=o}^{\infty} l_{n,r} = \psi_n(\tau_s) . \tag{3.24}$$

Thus,

$$L_n(t)e^{n\varepsilon} \leq \psi_n(\tau_s) \sum_{r=o}^{\infty} e^{2rt} \tag{3.25}$$

By the triangle inequality,

$$L_n(t)e^{n\varepsilon} - \psi_n(\tau_s) \leq \psi_n(\tau_s) \sum_{r=1}^{\infty} e^{2rt} . \tag{3.26}$$

Assuming

$$\psi_n(\tau) = e^{-nt}\, \Omega(\tau)H_n(t)e^{n\varepsilon} \tag{3.27}$$

results in equations similar to equations (3.27) through (3.26) with t substituted by $-t$ throughout. Thus, in general,

$$\psi_n(\tau) = \Omega(\tau)e^{n\varepsilon}[L_n(t)e^{nt}+H_n(t)e^{-nt}] \tag{3.28}$$

$$|L_n(t)e^{n\varepsilon}| - |\psi_n(\tau_s)| \leq |\psi_n(\tau_s) \sum_{r=1}^{\infty} e^{2rt}|$$

$$|H_n(t)e^{n\varepsilon}| - |\psi_n(\tau_s)| \leq |\psi_n(\tau_s) \sum_{r=1}^{\infty} e^{-2rt}| . \tag{3.29}$$

Equations (3.26) and (3.29) can be investigated more thoroughly with the following theorem.

Theorem 3 (Lighthill[3]). When $|\arg n| \leq \pi - \delta$,

$$\psi_n(\gamma-1/\gamma+1) \sim ke^{n\sigma}n^{1/6}$$

where $k = \pi B(1/3[\Gamma(2/3)]^{-1})2(2\gamma-1)/(2\gamma-2)(\gamma+1)-(\gamma+2)/(6\gamma-6)$.

When $|\arg(-n)| \leq \pi-\delta$, however,

$$\psi_n(\gamma-1/\gamma+1) \sim ke^{n\sigma}(-n)^{1/6} \sin(n\pi-\pi/6)/\sin(n\pi) .$$

Equations (3.26) and (3.29) hold only when $|2t| \leq |\sigma|$, otherwise the right hand sides are greater than $\psi_n(\tau_s)$. Thus for $|t|<|\sigma/2|$,

$$e^{n\varepsilon}L_n(t) = e^{n\varepsilon}H_n(t) \longrightarrow \psi_n(\tau_s) \ . \tag{3.30}$$

Theorem 4. For $|t|<|\sigma/2|$ and excluding circles of radius δ about the negative integers,

$$\psi_n(\tau) \sim ke^{n\sigma}|n|^{1/6}\Omega(\tau)[e^{nt}+e^{-nt}]X(n)$$

where $X_n=1$ when $|\arg n| \leq \pi-\delta$

$X_n = \sin(n\pi-\pi/6)/\sin(n\pi)$ when $|\arg(-n)| \leq \pi-\delta$ as $|n|\to\infty$.

The stipulation that $|t| \leq |\sigma/2| = .5867$ corresponds to $\tau = .5083$ and to a maximum Mach number of 2.2735 for $\gamma = 1.4$.

Chapter 4

THE INCOMPRESSIBLE-TO-COMPRESSIBLE TRANSFORMATION

Transformation in the Hodograph Plane

The series representing each $\psi_n(\tau)$ converges only in part of the hodograph plane. Bergman[4], Bers and Gelbart, and Lighthill[3] thus assume the total compressible stream function can be constructed from an infinite series of $\psi_n(\tau)$,

$$\psi = \mathrm{Im}\{\sum_{n=o}^{\infty} \psi_n(\tau)e^{-in\theta}\} \tag{4.1}$$

Consider an incompressible complex potential $\Phi^{(i)} = \phi^{(i)} + i\psi^{(i)}$ and define

$$\frac{d\Phi^{(i)}}{dz} = u - iv = qe^{-i\theta} \equiv \zeta \tag{4.2}$$

(see Appendix C). If such a complex potential is analytic outside the body, then in hodograph variables

$$\Phi^{(i)} = \sum_{n=o}^{\infty} c_n\zeta^n = \sum_{n=o}^{\infty} c_n q^n e^{-in\theta} \tag{4.3}$$

and

$$\psi^{(i)} = \mathrm{Im}\{\sum_{n=o}^{\infty} c_n q^n e^{-in\theta}\} . \tag{4.4}$$

The similarity of equations (4.1) and (4.3) prompted Lighthill to assume

$$\psi = \text{Im} \{ \sum_{n=o}^{\infty} c_n \psi_n(\tau) f(n, \tau_\infty) e^{-in\theta} \} \tag{4.5}$$

where $f(n,\tau_\infty)$ is called the normalizing function. Its primary duty is to force equation (4.4) to show the same behavior as equation (4.5) in the limit of vanishing compressibility.

In particular, for $o < \tau_\infty < \tau_s$ Lighthill stipulated:

(a) $f(n,\tau_\infty)$ is an analytic function of n except possibly at certain real non-negative poles of each of which it has a real residue.

(b) For large n $f(n,\tau_\infty) \sim Ae^{-ns_\infty}$ for some constant A.

(c) As $\tau_\infty \longrightarrow o$, $f(n,\tau_\infty) \sim \tau_\infty^{-n/2}$ uniformly for all n at a distance δ from any pole of $f(n,\tau_\infty)$.

Condition (b) ensures parallel behavior between equations (4.5) and (4.4) for large n and $\tau = \tau_\infty$. Equation (4.5) for large n and $o < \tau < \tau_s$ becomes:

$$\psi = \text{Im} \{ \sum_{n=o}^{\infty} c_n V(\tau_\infty) e^{-in\theta} \} \tag{4.6}$$

while equation (4.4) with $q = 1$ is

$$\psi^{(i)} = \text{Im} \{ \sum_{n=o}^{\infty} c_n e^{-in\theta} \} \tag{4.7}$$

Condition (c) ensures that equation (4.5) reduces to equation (4.4) as $q_m \longrightarrow \infty$. In this case

$$\psi_n \longrightarrow \tau^{n/2}[1+0(\tau)]$$

and $\psi_n(\tau)f(n,\tau_\infty)$ becomes $(\tau/\tau_\infty)^{n/2} = (q/q_\infty)^n = q^n$ for q_∞ normalized to one.

Lighthill considered closed contours about the origin of the complex n plane. For $\tau < \tau_s$, Lighthill postulated a function:

$$J_1(n,\tau,\tau_\infty) = 1/n(\psi_n(\tau)f(n,\tau_\infty)e^{n(s_\infty - s)} - f(o,\tau_\infty)) \ . \tag{4.9}$$

One notes:

(i) Because $\psi_o(\tau)f(o,\tau_\infty)e^o = f(o,\tau_\infty)$ a removable singularity exists at $n = o$.

(ii) By theorem 2, for large n, $J_1 \sim 1/n(AV(\tau)) \longrightarrow o$ as $|n| \longrightarrow \infty$.

(iii) By condition (a) and theorem 1, the poles $p_1, p_2, p_3, \ldots$ of $\psi_n(\tau)f(n,\tau_\infty)$ are real as are all the residues $r_1, r_2, r_3 \ldots$ of $n^{-1}\psi_n(\tau)f(n,\tau_\infty)$.

Lighthill postulated another function:

$$J_2(n,\tau,\tau_\infty) = \sum_{m=1}^{\infty} \frac{r_m e^{m(s_\infty - s)}}{n - p_m} . \tag{4.10}$$

One notes:

(iv) If $\sum_{m=1}^{\infty} r_m e^{p_m(s_\infty - s)}$ converges absolutely then J_2 is an analytic function of n except at the poles $p_1, p_2, p_3, \ldots$

(v) As $|n| \longrightarrow \infty$ and n remains a distance δ away from each p_m, $J_2 \longrightarrow 0$.

(vi) At each pole p_m the principal part of J_2 is the same as that of J_1.

Lighthill noted that the function $J_1 - J_2$ is entire in the n plane by observations (i), (iii), (iv), (vi). By observations (ii) and (v), its maximum modulus approaches 0 as $|n| \longrightarrow \infty$.

Thus:

$$\frac{1}{n}\left(\psi_n(\tau)f(n,\tau_\infty)e^{n(s_\infty - s)} - f(o,\tau_\infty)\right) = \sum_{m=1}^{\infty} \frac{r_m e^{p_m(s_\infty - s)}}{n - p_m} \tag{4.11}$$

$$\psi_n(\tau)f(n,\tau_\infty) = f(o,\tau_\infty)e^{n(s_\infty - s)} + n \sum_{m=1}^{\infty} \frac{r_m e^{(n-p_m)(s_\infty - s)}}{n-p_m} \tag{4.12}$$

and, as $|n| \rightarrow \infty$ yet remaining a distance δ from each p_m ,

$$AV(\tau) = f(o,\tau_\infty) + \sum_{m=1}^{\infty} r_m e^{p_m(s_\infty - s)} \tag{4.13}$$

Substitution of equation (4.12) into equation (4.5) yields

$$\psi = \mathrm{Im}\left[f(o,\tau_\infty) \sum_{n=o}^{\infty} c_n e^{n(s-s_\infty - i\theta)} + \sum_{m=1}^{\infty} r_m e^{-ip_m\theta} \sum_{n=o}^{\infty} nc_n \frac{e^{(n-p_m)(s-s_\infty - i\theta)}}{n-p_m} \right] \tag{4.14}$$

The form of the left-most summation suggest $\zeta = e^{s-s_\infty - i\theta}$ when compared with equation (4.4). Thus

$$\psi = \mathrm{Im}\left\{ f(o,\tau_\infty)\Phi^{(i)} + \sum_{m=1}^{\infty} r_m e^{-ip_m\theta} \int_{\zeta_o}^{e^{s-s_\infty - i\theta}} nc_n \zeta^{n-p_m-1} d\zeta + \sum_{n=1}^{\infty} nc_n \frac{\zeta_o^{n-p_m}}{n-p_m} \right\} \tag{4.15}$$

$$\psi = \mathrm{Im}\left\{ f(o,\tau_\infty)\Phi^{(i)} + \sum_{m=1}^{\infty} \left[r_m e^{-ip_m\theta} \int_{\zeta_o}^{e^{s-s_\infty - i\theta}} \zeta^{-p_m} d\Phi^{(i)} + g_m \right] \right\} \tag{4.16}$$

noting $d\Phi^{(i)} = \Sigma nc_n \zeta^{n-1} d\zeta$ from equation (4.3)

Lighthill noted that $g_m = f(m,\tau_\infty,\zeta_o)$. If the p_m is a pole of ψ_n , then $r_m = f(\psi_{-pm})$ by theorem 1. If the p_m is a pole of $f(n,\tau_\infty)$, then $r_m = f(\psi_{pm})$. Thus

$$\sum_{m=1}^{\infty} r_m g_m e^{-ip_m\theta} \tag{4.17}$$

is independently a solution of the equation of motion as is the remainder of equation (4.16), or

$$\psi = \mathrm{Im}\left\{ f(o,\tau_\infty)\Phi^{(i)} + \sum_{m=1}^{\infty} r_m e^{-ip_m\theta} \int_{\zeta_o}^{e^{s-s_\infty-i\theta}} \zeta^{-p_m} d\Phi^{(i)} \right\} . \tag{4.18}$$

Assuming $p_o = o$, $r_o = f(o,\tau_\infty)$, and taking the imaginary part of equation (4.18), yields:

$$\psi = \sum_{m=o}^{\infty} r_m \int_{q_o}^{e^{s-s_\infty}} q_1^{-p_m} \left(-\sin p_m(\theta-\theta_1) d\Phi_1^{(i)} + \cos p_m(\theta-\theta_1) d\Psi_1^{(i)}\right) \tag{4.19}$$

The subscript one denotes the variable of integration and ζ_o , $q_o \neq 0$ for $p_m \geq 1$ to ensure convergence of equation (4.18). The relation $\zeta = e^{s-s_\infty-i\theta}$ is the transformation from the incompressible flow plane to the compressible flow plane. Thus $q = e^{s-s_\infty}$ defines the speed magnitude transformation while the flow angle θ is invariant under the transformation.

Equation (4.14) transforms incompressible flow solutions to compressible flow solutions via simple numerical integration if the details of the governing Laplace solutions are known numerically.

An equation similar to equation (4.19) is now derived for supersonic flow. The procedure is exactly analogous to Lighthill's subsonic case. Note also, no constraints exist on $f(n,\tau_\infty)$ with respect to τ . Thus, any form of $f(n,\tau_\infty)$ valid for subsonic flow is also valid for supersonic flow.

where

Consider, for $0<|t|<|\sigma/2|$, $t>0$,

$$J_3(n,\tau,\tau_\infty)=\frac{1}{n^{7/6}}\left(\psi_n(\tau)f(n,\tau_\infty)e^{n(s_\infty-\sigma-t)}-f(o,\tau_\infty)\right) \tag{4.20}$$

Again, note:

(vii) A singularity of order 1/6 exists at $n = 0$.

(viii) By theorem 4 and condition (b) for $t>0$, $|J_3|\longrightarrow 0$ as $|n|\longrightarrow\infty$ while remaining a distance δ from any pole p_m .

(ix) Observation (iii) holds.

Consider, for $o<|t|<|\sigma/2|$, $t>o$,

$$J_4(n,\tau,\tau_\infty)=\frac{1}{n^{1/6}}\sum_{m=o}^{\infty} r_m \frac{e^{p_m(s_\infty-\sigma-t)}}{n-p_m} \ . \tag{4.21}$$

Note:

(x) A singularity of order 1/6 exists at $n = 0$.

(xi) r_m and p_m each include an additional value of 0 at $n = 0$.

(xiii) Observations (iv), (v), (vi) hold.

Thus J_3-J_4 is an entire function in the complex n plane whose modulus tends to zero as $|n|\longrightarrow\infty$ while avoiding the poles p_m .

Thus $J_3 = J_4$ and for $0<|t|<|\sigma/2|$, $t>0$

$$\psi_n(\tau)f(n,\tau_\infty)=f(o,\tau_\infty)e^{n(\sigma+t-s_\infty)}+n\sum_{m=o}^{\infty} r_m \frac{e^{(n-p_m)(\sigma+t-s_\infty)}}{n-p_m} \tag{4.22}$$

and as $|n|\longrightarrow\infty$ while remaining a distance δ from the p_m

$$|n|^{1/6}A\Omega(\tau)=f(o,\tau_\infty)+\sum_{m=o}^{\infty} r_m\, e^{p_m(s_\infty-\sigma-t)} \tag{4.23}$$

Equations (4.22) and (4.23) are exactly anologous to equations (4.12) and (4.13). Thus, by inspection, the form of equation (4.18) applicable to supersonic flow where $t > 0$ is,

$$\psi = \operatorname{Im} \; f(o,\tau_\infty)\zeta^{(i)} + \sum_{m=o}^{\infty} r_m e^{-ip_m\theta} \int_{q_o}^{e^{\sigma+t-s_\infty-i\theta}} \zeta^{-p_m}\, d\zeta^{(i)} \quad . \tag{4.24}$$

Letting $p_o = 0$, $r_o = f(o,\tau_\infty)$ and taking the imaginary part of the above gives

$$\psi = \sum_{m=o}^{\infty} r_m \int_{q_o}^{e^{\sigma+t-s_\infty}} q_1^{-p_m}\left(-\sin p_m(\theta-\theta_1)\,d\phi_1^{(i)} + \cos p_m(\theta-\theta_1)\,d\psi_1^{(i)}\right) \tag{4.25}$$

where one denotes a variable of integration. If $t < 0$, then substitute $+t$ into equations (4.20), (4.21), and (4.23) and $-t$ into equations (4.20) and (4.24). However, the result is the same in either case as $-(-t) = +t$. Equation (4.25) is therefore taken to be the supersonic analog to equation (4.19). Now, $q = e^{\sigma+t-s_\infty}$ and θ is still invariant under the transformation.

<u>Passage to the Physical Plane</u>

Derivation of the coordinates of the physical plane $x + iy = z$ begins with the Molenbroek-Chaplygin transformation,

$$Z_\theta = \frac{e^{i\theta}}{q}\left[\phi_\theta + i\left(\frac{\rho_o}{\rho}\right)\psi_\theta\right] \tag{4.26}$$

$$\phi_\theta = Q\psi_\theta \quad ; \quad Q = 2\tau(1-\tau)^{-1/\gamma-1} \tag{4.27}$$

$$Z_\theta = \frac{e^{i\theta}}{q}\left[\psi_\tau + i\left(\frac{\rho_o}{\rho Q}\right)\psi_o\right] \tag{4.28}$$

$$\rho_o/\rho=(1-\tau)^{-1/\gamma-1} \tag{4.29}$$

Thus, $$\rho_o/\rho Q = 1/2\tau \tag{4.30}$$

Recalling $$\frac{1}{q} = \frac{q_\infty}{q} = \frac{q_\infty/q_m}{q/q_m} = (\frac{\tau_\infty}{\tau})^{1/2} \tag{4.31}$$

Equation (4.28) becomes

$$Z_\theta = Q(\frac{\tau_\infty}{\tau})^{1/2} e^{i\theta} [\psi_\theta + \frac{i}{2\tau}\psi_\tau] \ . \tag{4.32}$$

The values of ψ_θ and ψ_τ are obtained from equations (4.19) and (4.25) for subsonic flow and supersonic flow, respectively. However, the only differeence between these two equations lies in the relationship between q and τ. Equations (4.19) and (4.25) yield exactly the same relationships when operated on by equation (4.32).

The remainder of this entire chapter is from Lighthill[3] with the exception of the generations to include supersonic flows.

$$\begin{aligned} Z_\theta = Q(\frac{\tau_\infty}{\tau})^{1/2} e^{i\theta} \Big\{ & \sum_{m=o}^{\infty} r'_m \int q_1^{-p_m} (-\sin p_m(\theta-\theta_1) d\phi_1^{(i)} + \cos p_m(\theta-\theta_1) d\psi_1^{(i)} \\ & + \sum_{m=o}^{\infty} r_m q^{-p_m} \psi_\tau + \frac{i}{2\tau} \sum_{m=o}^{\infty} r_m q^{-p_m} \psi_\tau \\ & + \frac{i}{2\tau} \sum_{m=o}^{\infty} r_m p_m \int q_1^{-p_m} (-\cos p_m(\theta-\theta_1) d\phi_1^{(i)} - \sin p_m(\theta-\theta_1) d\psi_1^{(i)}) \Big\} \ . \end{aligned} \tag{4.33}$$

Redefine $V(\tau)$, $|n|^{1/6}\Omega(\tau)$ as $\xi_n(\tau)$ to permit the development of a general passage to the physical plane, irrespective of flow character.

Note that $\Sigma\ r_m\, q^{-p_m} = A\xi_n(\tau)$ by equations (4.13) and (4.23) and by the incompressible-to-compressible speed magnitude relations. Next, use the facts

$$\frac{\partial}{\partial\theta}\left\{e^{i\theta}\int q_1^{-p_m}\left[\frac{e^{-ip_m(\theta-\theta_1)}}{2(p_m-1)}(d\phi_1^{(i)}+id\psi_1^{(i)})+\frac{e^{ip_m(\theta-\theta_1)}}{2(p_m+1)}(d\phi_1^{(i)}-id\psi_1^{(i)})\right]\right\}$$

$$=\int q_1^{-p_m}\left[-\frac{i}{2}\,e^{i\theta-ip_m(\theta-\theta_1)}(d\phi_1^{(i)}+id\psi_1^{(i)})+\frac{i}{2}\,e^{i\theta+ip_m(\theta-\theta_1)}(d\phi_1^{(i)}-id\psi_1^{(i)})\right]$$

$$+e^{i\theta}q^{-p_m}\left(\frac{1}{2(p_m-1)}\frac{\partial(\phi^{(i)}+i\psi^{(i)})}{\partial\theta}+\frac{1}{2(p_m+1)}\frac{\partial(\phi^{(i)}-i\psi^{(i)})}{\partial\theta}\right)$$

$$\frac{\partial}{\partial\theta}\left\{e^{i\theta}\int q_1^{-p_m}\left[\frac{e^{-ip_m(\theta-\theta_1)}}{2(p_m-1)}(d\psi_1^{(i)}-d\phi_1^{(i)})+\frac{e^{ip_m(\theta-\theta_1)}}{2(p_m+1)}(d\psi_1^{(i)}+d\phi_1^{(i)})\right]\right\} \tag{4.34}$$

$$=\int q_1^{-p_m}\left[-\frac{i}{2}e^{i\theta-ip_m(\theta-\theta_1)}(d\psi_1^{(i)}-id\phi_1^{(i)})+\frac{i}{2}e^{i\theta+ip_m(\theta-\theta_1)}(d\psi_1^{(i)}+id\phi_1^{(i)})\right]$$

$$+e^{i\theta}q_1^{-p_m}\left(\frac{1}{2(p_m-1)}\frac{\partial(\psi^{(i)}-i\phi^{(i)})}{\partial\theta}+\frac{1}{2(p_m+1)}\frac{\partial(\psi^{(i)}+i\phi^{(i)})}{\partial\theta}\right) \tag{4.35}$$

to reduce equation (4.33) to:

$$Z=Q\left(\frac{\tau_\infty}{\tau}\right)^{1/2}\sum_{m=o}^{\infty} r'_m e^{i\theta}\, q_1^{-p_m}\ \frac{e^{-ip_m(\theta-\theta_1)}}{2(p_m-1)}(d\phi_1^{(i)}+id\psi_1^{(i)})$$

$$+\frac{e^{ip_m(\theta-\theta_1)}}{2(p_m+1)}(d\phi_1^{(i)}-id\psi_1^{(i)})$$

$$+\frac{i}{2\tau}\sum_{m=o}^{\infty} r_m p_m e^{i\theta}\ q_1^{-p_m}\ \frac{e^{-ip_m(\theta-\theta_1)}}{2(p_m-1)}(d\psi_1^{(i)}-id\phi_1^{(i)})$$

$$+\frac{e^{ip_m(\theta-\theta_1)}}{2(p_m+1)}(d\psi_1^{(i)}+id\phi_1^{(i)}) \;+\; e^{i\theta}A\ \left(\phi_\tau+\frac{i}{2\tau}\psi_\theta\right)$$

$$- \sum_{m=o}^{\infty} r'_m \, q^{-p_m} \left(\frac{\phi_\theta^{(i)} + i \psi_\theta^{(i)}}{2(p_m - 1)} + \frac{\phi_\theta^{(i)} - i \psi_\theta^{(i)}}{2(p_m+1)} \right.$$

$$\left. - \frac{i}{2\tau} \sum_{m=o}^{\infty} r_m p_m q^{-p_m} \left(\frac{\psi_\theta^{(i)} - i \phi_\theta^{(i)}}{2(p_m-1)} + \frac{\psi_\theta^{(i)} + i \phi_\theta^{(i)}}{2(p_m+1)} \right) \right) d\theta + g(\tau) \; . \qquad (4.36)$$

Noting $\rho_o/\rho = 1$ in incompressible flow,

$$\psi_\tau^{(i)} = \frac{\partial \psi^{(i)}}{\partial \ln q} \cdot \frac{ds}{d\tau} = \psi_\theta^{(i)} \frac{ds}{d\tau} ; \quad \phi_\tau^{(i)} = \frac{\partial \phi^{(i)}}{\partial \ln q} \cdot \frac{dt}{d\tau} = \phi_\theta^{(i)} \frac{dt}{d\tau} \; . \qquad (4.37)$$

Also,

$$\int (\phi_\theta^{(i)} + i \psi_\theta^{(i)}) e^{i\theta} d\theta = q \int \frac{e^{i\theta}}{q} (d\phi^{(i)} + i d\psi^{(i)}) = q \int \zeta^{-1} d\zeta^{(i)} = qz \qquad (4.38)$$

$$\int (\phi_\theta^{(i)} - i \psi_\theta^{(i)}) e^{i\theta} d\theta = \frac{1}{q} \int q e^{i\theta} (d\phi^{(i)} - i d\psi^{(i)}) = \frac{1}{q} \int \bar{\zeta} \, \overline{d\zeta^{(i)}} = \frac{1}{q} \int \overline{\zeta^2 dz} \; . \qquad (4.39)$$

Let the incompressible-to-compressible transformation be denoted e^k where $k = s - s_\infty$ subsonically and $k = \sigma + t - s_\infty$ supersonically. Equation (4.36) becomes,

$$z = q \left(\frac{\tau_\infty}{\tau} \right)^{1/2} \sum_{m=o}^{\infty} \left(r'_m + \frac{r_m p_m}{2\tau} \right) \frac{e^{i(1-p_m)\theta}}{2(p_m - 1)} \int \zeta^{1-p_m} dz$$

$$+ \sum_{m=o}^{\infty} \left(r'_m - \frac{r_m p_m}{2\tau} \right) \frac{e^{i(1+p_m)\theta}}{2(1+p_m} \int \overline{\zeta^{1-p_m} dz}$$

$$A\zeta_n \frac{d(s,t)}{d} \cdot \frac{1}{2} \left(e^k z + e^{-k} \int \overline{\zeta^2 dz} \right) + \frac{i}{2} A\zeta_n \cdot \frac{1}{2i} \left(e^k z - e^{-k} \int \overline{\zeta^2 dz} \right)$$

$$- \sum_{m=o}^{\infty} r'_m e^{-p_m k} \left(\frac{e^k z}{2(p_m-1)} + \frac{e^{-k}}{2(p_m+1)} \int \overline{\zeta^2} dz \right.$$

$$- \frac{i}{2\tau} \sum_{m=o}^{\infty} r_m p_m e^{-p_m k} \left(\frac{-ie^k z}{2(p_m-1)} + \frac{ie^{-k}}{2(p_m+1)} \int \overline{\zeta^2} dz \right) + g(\tau) \tag{4.40}$$

The first line of equation (4.40) corresponds to the first and third terms of equation (4.36). The second line corresponds to the second and fourth terms while the third line corresponds to the sixth term and the last line corresponds to the last term.

Collecting the coefficients of Z yields $Q(\tau_\infty/\tau)^{1/2}$ times

$$\frac{1}{2} A\xi_n \frac{d(s,t)}{d\tau} e^k + \frac{1}{4\tau} A\xi_n e^k + \sum_{m=o}^{\infty} \left(r'_m + \frac{r_m p_m}{2\tau}\right) \frac{e^{(1-p_m)k}}{2(1-p_m)} \; . \tag{4.41}$$

Redefine equations (4.12) and (4.22) as

$$\psi_n(\tau) f(n,\tau_\infty) = f(o,\tau_\infty) e^{-nk} + n \sum_{m=1}^{\infty} r_m \frac{e^{(n-p_m)k}}{n-p_m} = n \sum_{m=o}^{\infty} r_m \frac{e^{(n-p_m)k}}{n-p_m} \tag{4.42}$$

and equations (4.13) and (4.23) as,

$$A\xi_n(\tau) = f(o,\tau_\infty) + \sum_{m=1}^{\infty} r_m e^{-p_m k} = \sum_{m=o}^{\infty} r_m e^{-p_m k} \tag{4.43}$$

where $r_o = f(o,\tau_\infty)$ and $p_o = 0$. Equation (4.42) with $n = 1$ is:

$$\psi_1(\tau) f(1,\tau_\infty) = \sum_{m=o}^{\infty} r_m \frac{e^{(1-p_m)k}}{1-p_m} \tag{4.44}$$

and

$$\psi_1'(\tau)f(1,\tau_\infty) = \sum_{m=o}^{\infty} r_m' \frac{e^{(1-p_m)k}}{1-p_m} + \frac{d(s,t)}{d\tau} \sum_{m=o}^{\infty} r_m e^{(1-p_m)k} \tag{4.45}$$

Using the above relations with equation (4.43), equation (4.41) reduces to:

$$\frac{1}{2}f(1,\tau_\infty)(\psi_1'(\tau)+ \frac{1}{2\tau}\psi_1(\tau)) \tag{4.46}$$

Similarly, the coefficient of $\zeta^2 dz$ is $Q(\tau_\infty/\tau)^{1/2}$ times

$$\frac{A\xi_n}{2}\frac{d(s,t)}{d} e^{-k} - \frac{1}{4}A\xi_n e^{-k} - \sum_{m=o}^{\infty} (r_m' - \frac{r_m p_m}{2}) \frac{e^{-(1+p_m)k}}{2(1+p_m)} \tag{4.47}$$

Equation (4.47) reduces to

$$-\frac{1}{2}f(-1,\tau_\infty)(\psi_{-1}'(\tau)+ \frac{1}{2\tau}\psi_{-1}(\tau)) \tag{4.48}$$

which, by the way of

$$\psi_{-1}(\tau)=\tau^{-1/2}+(2\gamma-2)^{-1}\psi_1(\tau) \tag{4.49}$$

further reduces to

$$\frac{\tau^{1/2}}{(\gamma-1)Q} . \tag{4.50}$$

Finally,

$$Z=\tau_\infty^{1/2}f(1,\tau_\infty)z - \frac{\tau_\infty^{1/2}f(-1,\tau_\infty)}{2\gamma-2}\int \zeta^2 dz +Q(\frac{\tau_\infty}{\tau})^{1/2} \Bigg[\sum_{m=o}^{\infty} (r_m'$$

$$+ \frac{r_m p_m}{2\tau}) \frac{e^{i(1-p_m)\theta}}{2(p_m-1)}\int \zeta^{1-p_m} dz$$

$$+ \sum_{m=o}^{\infty} (r_m' - \frac{r_m p_m}{2\tau}) \frac{e^{i(1+p_m)\theta}}{2(p_m+1)}\int \zeta^{(1-p_m)} dz \Bigg] + g(\tau) . \tag{4.51}$$

The arbitrary function of τ , $g(\tau)$, is found to reduce to an arbitrary constant upon operation under the counterpart to equation (4.32), namely,

$$Z_\tau = Q\left(\frac{\tau_\infty}{\tau}\right)^{1/2} e^{i\theta}\left(\frac{\psi_\theta^{(i)}}{PQ} + \frac{i}{2\tau}\psi_\tau^{(i)}\right) \tag{4.52}$$

This exercise is not followed. The reader is referred to Boerstoel's[5] concise treatment of the exercise.

Equation (4.51) was derived under the assumptions $p_m \neq 1, -1$. By theorem 1 and condition (a), no pole exists at $n = -1$. However, it is possible for $f(n,\tau_\infty)$ to have a pole at $n = +1$. A new equation giving the correspondence between the hodograph plane and physical plane and valid for $p_m = 1$ must be deduced. The process is very similar to the one given above and the reader is referenced to Lighthill[3] or Boerstoel[5] for the treatment.

$$Z = Q\left(\frac{\tau_\infty}{\tau}\right)^{1/2}\Big[Z \lim_{n\to 1}\left(\frac{1}{2}\left(\psi_n'(\tau)f(n,\tau_\infty) - \frac{r_1'}{n-1}\right) + \frac{1}{4\tau}\left(\psi_n(\tau)f(n,\tau_\infty) - \frac{r_1}{n-1}\right)\right]$$
$$- \frac{\tau^{1/2}f(-1,\tau_\infty)}{(2\gamma-2)Q}\int \overline{\zeta^2 dz} - \frac{1}{2}\left(r_1' + \frac{r_1}{2\tau}\right)\left(\int \ln\zeta dz + i\theta z\right) + \sum_{m=2}^{\infty}\left(r_m'\right.$$
$$\left. + \frac{r_m p_m}{2\tau}\right)\frac{e^{i(1-p_m)\theta}}{2(p_m-1)}\int \zeta^{1-p_m}dz + \sum_{m=2}^{\infty}\left(r_m' - \frac{r_m p_m}{2\tau}\right)\frac{e^{i(1+p_m)\theta}}{2(p_m+1)}\int \zeta^{1-p_m}dz\Big] \tag{4.53}$$

<u>Normalization Functions</u>

The actual form(s) of $f(n,\tau_\infty)$ must be found satisfying not only conditions (a), (b), and (c) but also ensuring that

$$\sum_{m=1}^{\infty} r_m e^{-p_m k} \tag{4.54}$$

converges absolutely. An obvious choise is

$$f(n,\tau_\infty) = e^{-ns_\infty} \tag{4.55}$$

given by condition (b). Conditions (a) and (c) are also satisfied. Because no poles exist, the convergence of equation (4.54) is determined by the r_m due to the $\psi_n(\tau)$'s . By theorem 1, convergence is assured if $(\sigma+s_\infty-s)<0$ for subsonic flow and if $(s_\infty-t)<0$ for the supersonic case. The subsonic case is always true. The supersonic case is true beyond the range of validity of theroem 4 if the free stream is subsonic. Because $n = 1$ is not a pole, equation (4.51) may be used for passage back to the physical plane.

Recall that the hodograph surface possesses a singularity at $\zeta = 1$ which corresponds to the free stream conditions. Thus, when circulation is present, the integrals of ζ to any power with the variable of integration as Z will all suffer a fixed increase once this free stream singularity is encircled.

Derivation of the normalization function for circulatory flow depends on forcing the said integrals to remain one valued. Consider equation (4.16) and note

$$d\Phi^{(i)} = \zeta dZ \ . \tag{4.56}$$

The integral in equation (4.16) becomes $\zeta^{1-p_m}\,dz$. For ζ near one,

$$\zeta^{1-p_m}=1+(1-p_m)(\zeta-1)+0|\zeta-1|^2 \tag{4.57}$$

The integral $\oint p_m dz$ is zero while the integral $\oint \zeta dz$ is simply the circulation, Γ . Thus, the integral $\oint \zeta^{1-p_m}\,dz$ suffers a fixed increase of $(1-p_m)\Gamma$ as the free stream singularity is encircled. Therefore, equation (4.16) and (4.25) increased by

$$-\Gamma \sum_{m=1}^{\infty} (1-p_m)c_m \sin p_m \theta \ . \tag{4.58}$$

This sum can be zero only if the p_m's are symmetrically placed about the origin in the n plane (except possibly $p_m = 1$ which needs no counterpart). Denoting these poles as $p_{m'} = -p_m$, the residues must satisfy

$$r'_m = \frac{1-p_m}{1+p_m} r_m \tag{4.59}$$

The residue of $n^{-1}\psi_n(\tau)f(n,\tau_\infty)$ due to $\psi_n(\tau)$ at $n = -m$ (for $m \geq 2$) is

$$C_m\psi_m(\tau)f(-m,\tau_\infty) \ . \tag{4.60}$$

By equation (4.59) that at $n = m$ must be

$$(1+m/1-m)C_m\psi_m(\tau)f(-m,\tau_\infty) \tag{4.61}$$

so that the residue of $f(n,\tau_\infty)$ at $n = m$ is:

$$m(1+m/1-m)C_m f(-m,\tau_\infty) \tag{4.62}$$

Lighthill next found the form of $f(n,\tau_\infty)$ which satisfies the above as well as conditions (a), (b), and (c) through educated guesses. It is:

$$f(n,\tau_\infty) = \frac{\psi_{-n}(\tau_\infty)+2\tau_\infty\psi'_{-n}(\tau_\infty)}{(1-n)} \tag{4.63}$$

A pole exists at $n = 1$. Equation (4.53) should also be one valued with this form of $f(n,\tau_\infty)$. As the free stream singularity is encircled, equation (4.53) increase by:

$$Q\left(\frac{\tau_\infty}{\tau}\right)^{1/2}\Gamma\left[\frac{-(\tau)^{1/2}f(-1,\tau_\infty)}{(\gamma-1)Q} + \sum_{m=2}^{\infty}\left[\left(r'_m + \frac{r_m p_m}{2\tau}\right)\frac{e^{i(1-p_m)\theta}}{-2} + \right.\right.$$
$$\left.\left. +\left(r'_m - \frac{r_m p_m}{2\tau}\right)\frac{e^{i(1+p_m)\theta}}{2}\left(\frac{1-p_m}{1+p_m}\right)\right] - \frac{1}{2}\left(r'_1 + \frac{r_1}{2\tau}\right)\right] \tag{4.64}$$

By equation (4.59) the summation may be recast as

$$\sum_{r=2}^{\infty}\left[\left(r_m' + \frac{r_m p_m}{2\tau}\right)\frac{e^{i(1-p_m)\phi}}{-2} + \left(r_m' + \frac{r_m' p_m'}{2\tau}\right)\frac{e^{i(1-p_m')\phi}}{2}\right] \quad (4.65)$$

which is zero by the symmetry about $n = 0$. Equation (4.53) is one-valued only if

$$r_1' + \frac{r_1}{2} + \frac{2\tau^{1/2}}{(\gamma-1)Q} f(-1,\tau_\infty) = 0 \quad (4.66)$$

By equation (4.63).

$$f(-1,\tau_\infty) = \frac{1}{2}[\psi_1(\tau_\infty) + 2\tau_\infty\psi_1'(\tau_\infty)] \quad (4.67)$$

$$r_1 = -\psi_1(\tau)[\psi_{-1}(\tau_\infty) + 2\tau_\infty\psi_{-1}'(\tau_\infty)) \quad (4.68)$$

$$r_1' = -\psi_1'(\tau)[\psi_{-1}(\tau_\infty) + 2\tau_\infty\psi_{-1}'(\tau_\infty)] \quad (4.69)$$

But

$$\psi_1(\tau) = (\gamma-1)\ \{1-(1-\tau)^{\gamma/\gamma-1}\}/\gamma\tau^{1/2} \quad (4.70)$$

$$\psi_1'(\tau) = \frac{-1}{2\tau^{3/2}}\ (\gamma-1)\{1-(1-\tau)^{\gamma/\gamma-1}\}/\gamma + (1-\tau)^{1/\gamma-1}/\tau^{1/2} \quad (4.71)$$

Substituting all of the above into equation (4.66) gives

and,
$$\psi_{-1}(\tau_\infty) + 2\tau_\infty\psi_{-1}'(\tau_\infty) = \frac{1}{2(\gamma-1)}[\psi_1(\tau_\infty) + 2\tau_\infty\psi_1'(\tau_\infty)]$$

$$\frac{(1-\tau_\infty)^{1/\gamma-1}}{\gamma-1}\tau_\infty^{1/2} = \frac{(1-\tau_\infty)^{1/\gamma-1}}{\gamma-1}\tau_\infty^{1/2} \ . \quad (4.72)$$

Thus, equation (4.66) does indeed equal zero and one can state conclusively that

$$f(n,\tau_\infty) = \frac{\psi_{-n}(\tau_\infty) + 2\tau_\infty\psi_{-n}'(\tau_\infty)}{1-n} \quad (4.73)$$

is a proper normalizing function for circulatory flows.

At the poles:

(i) $n = -m(m-2,3,4,\ldots)$, the residue of $n^{-1}\psi_n(\tau)f(n,\tau_\infty)$ is

$$C_m\psi_m(\tau)[\psi_m(\tau_\infty)+2\tau_\infty\psi'_m(\tau_\infty)](1+m) \qquad (4.74)$$

(ii) $n=m(m=2,3,4,\ldots)$, the residue is

$$C_m\psi_m(\tau)[\psi_m(\tau_\infty)+2\tau_\infty\psi'_m(\tau_\infty)]/(1/m) \qquad (4.75)$$

(iii) $n = 1$, the residue is

$$-\psi_1(\tau)[\psi_{-1}(\tau_\infty)+2\tau_\infty\psi'_{-1}(\tau_\infty)] \qquad (4.76)$$

Lighthill[3] stated that equation (4.53) becomes

$$Z=F(\tau,\tau_\infty)Z+\frac{\tau_\infty(1-\tau_\infty)^{1/\gamma-1}}{\gamma-1}\,[(\ln\zeta\,dZ+i\theta Z-\frac{1}{2}\int\overline{\zeta^2 dZ}$$

$$-\frac{\tau+(\gamma-1)(1-(1-\tau)^{-1/\gamma-1}}{2\gamma\tau}\,Ze^{2i\theta}]+Q(\frac{\tau_\infty}{\tau})^{1/2}\sum_{m=2}^{\infty}C_m(\psi_m(\tau_\infty)$$

$$+2\tau_\infty\psi'_m(\tau_\infty))\{(\psi'_m(\tau)-\frac{m}{2\tau}\psi_m(\tau))\,\frac{e^{i(m+1)\theta}}{2(m+1)}\,(\overline{\int\frac{\zeta^{1-m}}{1-m}\,dZ}$$

$$-\int\frac{\zeta^{m+1}}{m+1}\,dz)\;+(\psi'_m(\tau)+\frac{m}{2\tau}\psi_m(\tau))\,\frac{e^{i(1-m)\theta}}{2(1-m}\,(\overline{\int\frac{\zeta^{m+1}}{m+1}\,dZ}$$

$$-\int\frac{\zeta^{1-m}}{1-m}\,dZ)\} \qquad (4.77)$$

where

$$F(\tau,\tau_\infty)=1+\frac{1}{2\gamma}\,(1-(1-\tau_\infty)^{\gamma/\gamma-1})+\frac{\tau_\infty(1-\tau_\infty)^{1/\gamma-1}}{\gamma-1}[\frac{1}{2}\ln|\frac{\tau_\infty}{\tau}|-$$

$$+(1+\frac{1}{2\gamma})\int_0^{\tau_\infty}\frac{(1-x)^{-\gamma/\gamma-1}-1}{x}\,dx+\frac{1}{2\gamma}\int_0^{\tau}\frac{(1-x)^{-\gamma/\gamma-1}-1}{x}\,dx] \; . \qquad (4.78)$$

Finally, by theorem 1, the convergence of equation (4.54) is guaranteed if $(2\sigma-s+s_\infty)<0$ for subsonic flow. This is never violated. By theorem 4, convergence is secured if $(\sigma+s_\infty-t)<0$ which is again valid beyond the range of validity of theorem 4. Thus, the assumption of convergence required in the transformation is valid under these two choices of normalizing functions.

The reader is again referred to Boerstoel[5] who derives several several other nornmalizing functions. They are, however, less general than Lighthill's and are applicable to simplistic model flows only.

Chapter 5

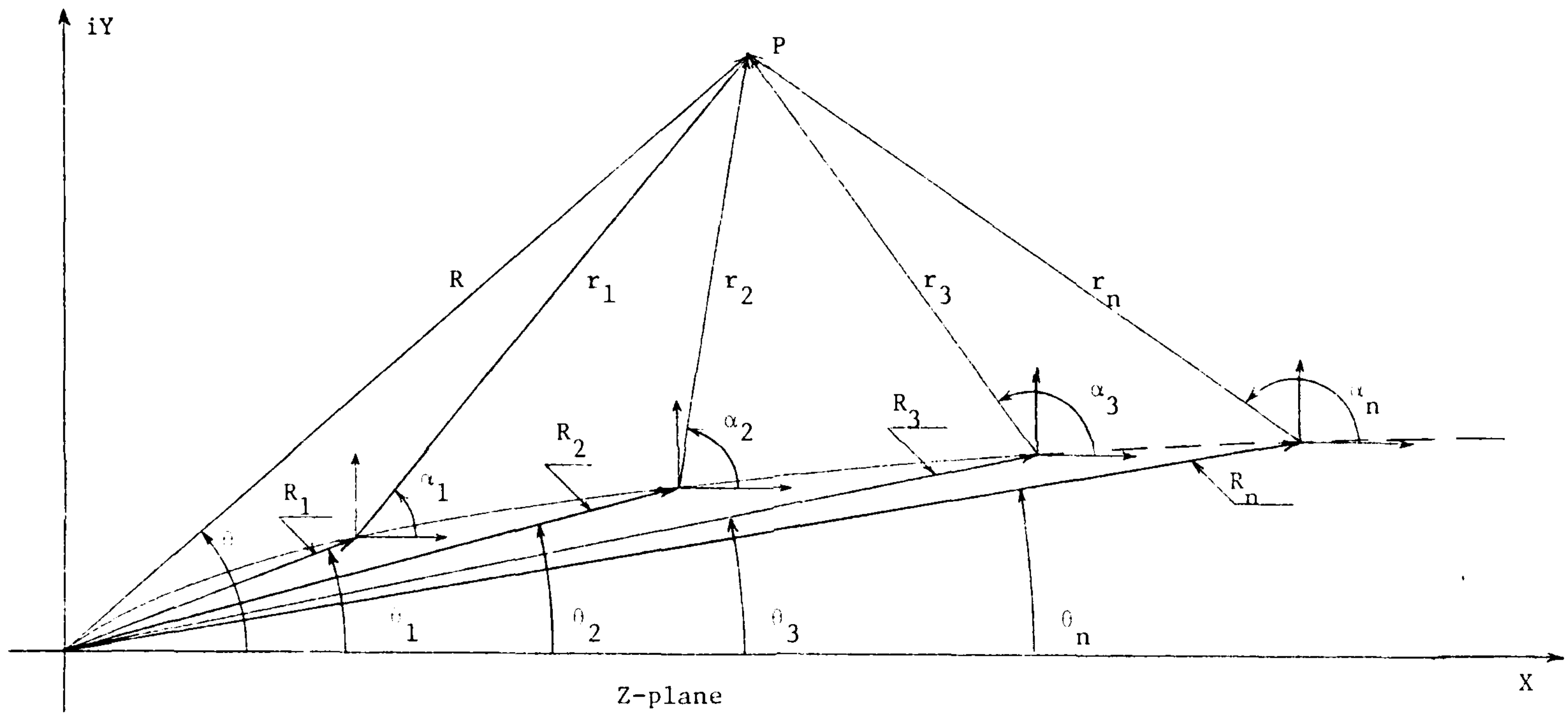
CONSTRUCTION OF THE MODEL FLOW AND THE VELOCITY FIELD

Construction of the model flow, or a governing incompressible complex potential, is required because Lighthill's transformation technique requires a detailed numerical knowledge of the governing imcompressible complex potential and the associated velocity field.

The coordinates of the profile, $\psi^{(i)} = 0$, to be transformed are assumed to be known. Also, it is assumed that a combination of doublets and clockwise vortices placed in a uniform flow represent the profile. The strengths and positions of the doublets and vortices are to be found through the knowledge of the given profile geometry. In what is to follow, it is assumed that the doublet centers and vortex centers are by no means coincident.

Consider the complex Z-plane in which the doublet and vortex centers are placed. See Figure 8. Assume a z_j coordinate center is placed at the center of the j^{th} singularity - be it a doublet or a vortex. Then, at a general point P, the incompressible complex potential is:

FIGURE 1



$$\Phi^{(i)} = \phi^{(i)} + i\psi^{(i)} = q_\infty Z + \sum_{a=1}^{n} \frac{B'_a}{Z_a} + \frac{i}{2\pi} \sum_{b=1}^{m} \Gamma'_b \ln|z_b| \tag{5.1}$$

Setting $q_\infty = 1$ and normalizing the doublet and vortex strengths yields:

$$\Phi^{(i)} = \phi^{(i)} + i\psi^{(i)} = Z + \sum_{a=1}^{n} \frac{B_a}{Z_a} + \frac{i}{2\pi} \sum_{b=1}^{m} \Gamma_b \ln|z_b| \tag{5.2}$$

Note that the j^{th} singularity center coordinates are $Z_j = R_j e^{i\theta j}$ in the Z-plane. The coordinates in the j^{th} Z-plane are $z_j = r_j e^{i\alpha j}$. Taking the real and imaginary parts of the incompressible complex potential gives:

$$\phi^{(i)} = R\cos\theta + \sum_{a=1}^{n} \frac{B_a \cos\alpha_a}{r_a} - \frac{1}{2\pi} \sum_{b=1}^{m} \Gamma_b \alpha_b \tag{5.3}$$

$$\psi^{(i)} = R\sin\theta - \sum_{a=1}^{n} \frac{B_a \sin\alpha_a}{r_a} + \frac{1}{2\pi} \sum_{b=1}^{m} \Gamma_b \ln|r_b| \tag{5.4}$$

Noting:

$$R\sin\theta = r_j \sin\theta_j + r_j \sin\alpha_j$$

$$R\cos\theta = r_j \cos\theta_j + r_j \cos\alpha_j$$

$$r_j^2 = R^2 + R_j^2 - 2RR_j \cos(\theta - \theta_j)$$

One may write:

$$\phi^{(i)} = R\cos\theta + \sum_{a=1}^{n} \frac{B_a(R\cos\theta - {}_dR_a \cos {}_d\theta_a)}{R^2 + ({}_dR_a)^2 - 2R\,{}_dR_a \cos(\theta - {}_d\theta_a)}$$

$$- \frac{1}{2\pi} \sum_{b=1}^{m} \Gamma_b \cos^{-1}\left[\frac{R\cos\theta - {}_vR_b \cos {}_v\theta_b}{R^2 + ({}_vR_b)^2 - 2R\,{}_vR_b \cos(\theta - {}_v\theta_b)}\right] \tag{5.5}$$

$$\psi^{(i)} = R\sin\theta + \sum_{a=1}^{n} \frac{B_a({}_dR_a \sin {}_d\theta_a - R\sin\theta)}{R^2 + ({}_dR_a)^2 - 2R\,{}_dR_a \cos(\theta - {}_d\theta_a)}$$

$$+ \frac{1}{4\pi} \sum_{b=1}^{m} \Gamma_b \ln|R^2 + ({}_vR_b^2) - 2R\,{}_vR_b \cos(\theta - {}_v\theta_b)| \tag{5.6}$$

where the subscripts d and v denote quantities due to doublets and vortices, respectively.

Now B_a, ${}_dR_a$, ${}_d\theta_a$, Γ_b, ${}_vR_b$, and ${}_v\theta_b$ must be determined. For n doublets and m vortices, there are 3(n+m) unknowns. Thus, specifying 3(n+m) coorindate points (R,θ) of the profile $\psi^{(i)} = 0$, results in a system of (3n+m) nonlinear equations in 3(n+m) unknowns. Such a system may be solved by a nonlinear counterpart of the Gauss-Seidel iteration method.

If 3(n+m) coordinate points of $\psi^{(i)} = 0$ are known, the one may set up the following system:

$$
\begin{aligned}
0 &= f_1(X_1, X_2, \ldots, X_{3(n+m)}) \\
0 &= f_2(X_1, X_2, \ldots, X_{3(n+m)}) \\
&\vdots \\
0 &= f_{3(n+m)}(X_1, X_2, \ldots, X_{3(n+m)})
\end{aligned}
$$

Letting:

$$
\begin{array}{lcl}
X_1 \longrightarrow X_n & \text{be} & B_1 \longrightarrow B_n \\
X_{n+1} \longrightarrow X_{n+m} & \text{be} & \Gamma_1 \longrightarrow \Gamma_m \\
X_{n+m+1} \longrightarrow X_{2n+m} & \text{be} & {}_dR_1 \longrightarrow {}_dR_n \\
X_{2n+m+1} \longrightarrow X_{2(n+m)} & \text{be} & {}_vR_1 \longrightarrow {}_vR_m \\
X_{2(n+m)+1} \longrightarrow X_{2(n+m)} & \text{be} & {}_d\theta_1 \longrightarrow {}_d\theta_n \\
X_{3n+2m+1} \longrightarrow X_{3(n+m)} & \text{be} & {}_v\theta_1 \longrightarrow {}_v\theta_m
\end{array}
$$

and letting superscripts k and k+1 denote successive iteration steps, superscript k^* denote the value after the k^{th} step unless the value of step $k+1$ is available, and subscript ℓ denote the ℓ^{th} given coordinate point of $\psi^{(i)} = 0$. The following iteration equations result. See following pages.

There are two healthy advantages to this Gauss-Seidel iteration method. First, truncation and round-off errors do not accumulate as they do in elimination methods. Each new value is, in a sense, a new initial guess. Second, a relatively small amount of memory is required. With N total unknown and N specified coordinate points, only 2N memory locations are needed. Contrast this to some elimination methods which require N^2 memory locations.

However, due to the strongly non-linear nature of the equations to be solved, one must carefully formulate an initial guess. The following procedure is suggested:

1. Build a symmetrical profile with a thickness distribution roughly equivalent to the desired non-symmetrical profile. Thus, $\Gamma_j = {}_vR_j = {}_v\theta_j = {}_d\theta_j = 0$.
2. Solve the first and third equations of the above iteration equations subject to step 1. Make initial guesses:

   $B_j^{\,o}$ = (Thickness at j)/n

   ${}_dR_j^{\,o}$ = (j/n) (chord length)
3. Change the symmetrical profile to a slightly non-symmetrical one.
4. Use the B_j and ${}_dR_j$ found by step 2 as an initial guess

$$1 \leq \ell \leq n$$

$$X_\ell^{k+1} \equiv B_j^{k+1} = \left\{ \frac{R_\ell^2 + ({}_dR_j^k)^2 - 2R_\ell\,{}_dR_j^k \cos(\theta_\ell - {}_d\theta_j^k)}{R_\ell \cos\theta_\ell - {}_dR_j^k \cos {}_d\theta_j^k} \right\} \left\{ R_\ell \sin\theta_\ell \right.$$

$$+ \sum_{\substack{a=1 \\ a \neq j}}^{n} \frac{B_a^{k*}({}_dR_a^k \sin {}_d\theta_a^k - R_\ell \sin\theta_\ell)}{R_\ell^2 + ({}_dR_a^k)^2 - 2R_\ell\,{}_dR_a^k \cos(\theta_\ell - {}_d\theta_a^k)}$$

$$\left. + \frac{1}{4\pi} \sum_{b=1}^{m} \Gamma_b^k \ell n \left| R_\ell^2 + ({}_vR_b^k)^2 - 2R_\ell\,{}_vR_b^k \cos(\theta_\ell - {}_v\theta_b^k) \right. \right\} \tag{5.7}$$

$$n+1 \leq \ell \leq n+m$$

$$X_\ell^{k+1} \equiv \Gamma_j^{k+1} = \left\{ \frac{-4\pi}{\ell n \left| R_\ell^2 + ({}_vR_j^b)^2 - 2R_\ell\,{}_vR_j^k \cos(\theta_\ell - {}_v\theta_j^k) \right.} \right\} \left\{ R_\ell \sin\theta_\ell \right.$$

$$+ \sum_{a=1}^{n} \frac{B_a^{k+1}({}_dR_a^k \sin {}_d\theta_a^k - R_\ell \sin\theta_\ell)}{R^2 + ({}_dR_a^k)^2 - 2R_\ell\,{}_dR_a^k \cos(\theta_\ell - {}_d\theta_a^k)}$$

$$\left. + \frac{1}{4\pi} \sum_{\substack{b=1 \\ b \neq j}}^{m} \Gamma_b^{k*} \ell n \left| R_\ell^2 + ({}_vR_b^k)^2 - 2R_\ell\,{}_vR_b^k \cos(\theta_\ell - {}_v\theta_b^k) \right. \right\} \tag{5.8}$$

$n+m+1 \le \ell \le 2n+m$

$$X_\ell^{k+1} \equiv {}_dR_J^{k+1} = \left\{ \frac{R_\ell^2 + ({}_dR_j^k)^2 - 2R_{\ell d}R_j^k \cos(\theta_\ell - {}_d\theta_j^k)}{-B_j^{k+1} \sin\theta_j^k} \right\} \left\{ R_\ell \sin\theta_\ell \right.$$

$$- \frac{B_j^{k+1} R_\ell \sin\theta_\ell}{R_\ell^2 + ({}_dR_j^k)^2 - 2R_{\ell d}R_j^k \cos(\theta_\ell - {}_d\theta_j^k)}$$

$$+ \sum_{\substack{a=1 \\ a \neq j}}^{n} \frac{B_a^{k+1}({}_dR_a^{k*} \sin {}_d\theta_a^k - R_\ell \sin\theta_\ell)}{R_\ell^2 + ({}_dR_a^{k*})^2 - 2R_{\ell d}R_a^{k*} \cos(\theta_\ell - {}_d\theta_a^k)}$$

$$\left. + \frac{1}{4\pi} \sum_{b=1}^{m} \Gamma_b^{k+1} \ln \left| R_\ell^2 + ({}_vR_b^k)^2 - 2R_{\ell v}R_b^k \cos(\theta_\ell - {}_v\theta_b^k) \right| \right\} \tag{5.9}$$

$2n+m+1 \le \ell \le 2(n+m)$

$$X_\ell^{k+1} \equiv {}_vR_j^{k+1} = \left\{ 2R_{\ell v}R_j^k \cos(\theta_\ell - {}_v\theta_j^k) - R_\ell^2 + \mathrm{EXP}\left[\frac{-4\pi}{\Gamma_j^{k+1}} \left[R_\ell \sin\theta_\ell \right.\right.\right.$$

$$+ \sum_{a=1}^{n} \frac{B_a^{k+1}({}_dR_a^{k+1} \sin {}_d\theta_a^k - R_\ell \sin\theta_\ell)}{R_\ell^2 + ({}_dR_a^{k+1})^2 - 2R_{\ell d}R_a^{k+1} \cos(\theta_\ell - {}_d\theta_a^k)}$$

$$\left.\left.\left. + \frac{1}{4\pi} \sum_{\substack{b=1 \\ b \neq j}}^{m} \Gamma_b^{k+1} \ln \left| R_\ell^2 + ({}_vR_b^{k*})^2 - 2R_{\ell v}R_b^{k*} \cos(\theta_\ell - {}_v\theta_b^k) \right| \right]\right] \right\}^{1/2} \tag{5.10}$$

$$2n+2m+1 \leq \ell \leq 3n+2m$$

$$X_\ell^{k+1} \equiv {}_d\theta_j^{k+1} = \sin^{-1}\left\{\left\{\frac{R_\ell^2+({}_dR_j^{k+1})^2-2R_{\ell d}R_j^{k+1}\cos(\theta_\ell-{}_d\theta_j^k)}{-B_j^{k+1}\ {}_dR_j^{k+1}}\right\}\left\{R_\ell \sin\theta_\ell\right.\right.$$

$$-\frac{B_j^{k+1}R_\ell \sin\theta_\ell}{R_\ell^2+({}_dR_j^{k+1})^2-2R_{\ell d}R_j^{k+1}\cos(\theta_\ell-{}_d\theta_j^{k*})}$$

$$+\sum_{\substack{a=1\\a\neq j}}^{n}\frac{B_a^{k+1}({}_dR_a^{k+1}\sin{}_d\theta_a^{k*}-R_\ell\sin\theta_\ell)}{R_\ell^2+({}_dR_a^{k+1})^2-2R_{\ell d}R_a^{k+1}\cos(\theta_\ell-{}_d\theta_a^{k*})}$$

$$\left.\left.+\frac{1}{4\pi}\sum_{b=1}^{m}\Gamma_b^{k+1}\ell n\left|R_\ell^2+({}_vR_b^{k+1})^2-2R_{\ell v}R_b^{k+1}\cos(\theta_\ell-{}_v\theta_b^k)\right|\right\}\right\} \tag{5.11}$$

$$3n+2m+1 \leq \ell \leq 3(n+m)$$

$$X_\ell^{k+1} \equiv {}_v\theta_j^{k+1} = \theta_\ell - \cos^{-1}\left[\frac{1}{2R_{\ell v}R_j^{k+1}}\left[R_\ell^2+({}_vR_j^{k+1})^2-\mathrm{EXP}\left[\frac{-4\pi}{\Gamma_j^{k+1}}\left[R_\ell\sin\theta_\ell\right.\right.\right.\right.$$

$$+\sum_{a=1}^{n}\frac{B_a^{k+1}({}_dR_a^{k+1}\sin{}_d\theta_a^{k+1}-R_\ell\sin\theta_\ell)}{R_\ell^2+({}_dR_a^{k+1})^2-2R_{\ell d}R_a^{k+1}\cos(\theta_\ell-{}_d\theta_a^{k+1})}$$

$$\left.\left.\left.\left.+\frac{1}{4\pi}\sum_{\substack{b=1\\b\neq j}}^{m}\Gamma_b^{k+1}\ell n\left|R_\ell^2+({}_vR_b^{k+1})^2-2R_{\ell v}R_b^{k+1}\cos(\theta_\ell-{}_v\theta_b^{k*})\right|\right]\right]\right]\right] \tag{5.12}$$

along with

$$d^{\theta}{}_j = v^{\theta}{}_j = 0$$

$$v^{R}{}_j = (j/m) \text{ (chord length)}$$

$$\Gamma_j^{\,O} = .01 \sin(j\pi/m)$$

5. Solve the above iteration equations subject to step 4.

6. Repeat steps 3 through 5 until the desired profile is obtained.

During the initial design stages one is interested in finding a rough profile with desirable flow characteristics. It is economical, therefore, to specify fewer coordinate points in this stage than in later stages where the precise profile is required. The above procedure need only be employed in the initial stage, however. As one specifies more coordinate points, one will have to specify additional initial guesses for additional doublets and vortices. These additional initial guesses may be intelligently chosen by interpolation of the final values yielded by the above procedure.

Leading and trailing edge closure will always be a problem with a finite number of doublets and vortices. Luckily, the non-closures may be reduced to negligible magnitudes by specifying closely spaced coordinate points at the nose and tail of the profile. The non-closure is thereby forced to fall between a particular pair of arbitrarily close points.

The angle of attack is easily found as the angle between the chord line and the normal to the profile at the front stagnation point.

The velocity field data are now at hand. Specification of any two points in the field allows the calculation of $\Delta\phi^{(i)}$ and $\Delta\psi^{(i)}$ by equations (5.5) and (5.6). If the two points are sufficiently close to each other, then

$$\Delta\phi^{(i)}=d\phi^{(i)}= udx + vdy \tag{5.13}$$

$$\Delta\psi^{(i)}=d\psi^{(i)}= -vdx+udy \tag{5.14}$$

Solution of equations (5.13) and (5.14) yields u and v. By

$$q = \sqrt{u^2+v^2} \tag{5.15}$$

$$\theta = \tan^{-1}(v/u) \tag{5.16}$$

The required data are known throughout the field.

Chapter 6

DISCUSSION

The hodograph method presented in Chapter 4 extends Lighthill's transformation method, valid for subsonic flows only, to transonic flows. The ability to treat supersonic regions springs from theorem 4.

An underlying assumption in theorem 4 is that τ is a function of $e^{\pm 2t}$. This assumption predicts that τ and t vary together, i.e. they both increase or they both decrease. Lighthill, however, assumed τ to be a function of $e^{\pm 2t}$ in his development of the asymptotic formulae of the Chaplygin functions for supersonic flow. This assumption is physically insensible as it predicts τ is constant in supersonic flow. Hence, the ratio $(q/q_m)^2$ is constant irrespective of t.

Consulting Lighthill's[3] work shows the form of theorem 4 is similar to Lighthill's analogous result. Both predict poles at negative integral n as well as the oscillatory behavior of $\psi_n(\tau)$ along the positive real axis when $\tau > \tau_s$. Both results predict that for large, positive, real n the zeroes of $\psi_n(\tau)$ are given approximately by the zeroes of the trigonometrical functions. Theorem 4, however, is far less complicated than Lighthill's analogous formulae. From this simplicity rises the transformation technique developed in this study which is markedly similar to Lighthill's method but is able to treat transonic flows.

All the advantages of Lighthill's technique are possessed by this newly developed method. In particular, any flow which can be represented by potential and stream functions may be transformed utilizing purely numerical data concerning the flow velocity magnitude and direction. No longer must one be restricted to those flows which can be represented as Laurent series expansions or Mellin-Barnes integrals in the hodograph plane. The governing incompressible complex potential about a general lifting profile is generally far too complicated to be represented thusly and cannot be employed as an incompressible boundary condition. The new method, however, is able to transform such a boundary condition.

This method eliminates the closure problem experienced by those who use such representations of the model flow in the hodograph plane. Previous researchers using hodograph methods to solve the compressible transonic flow about a profile have overlooked another work due to Lighthill[11]. Lighthill addressed the problem of inverse design and formulated three constraints which must be satisfied if the inverse problem is to be well-posed. They are:

where $q_o(\omega)$ is the prescribed speed distribution as a function of the polar angle of the circle obtained from a conformal mapping of the airfoil in question. Qualitatively, the first constraint states that the pressure distribution over the airfoil and the free stream speed cannot be specified independently of each other. The other two state that the specified pressure distribution and the angle of attack may not be independently specified and that the trailing edge close. Trailing edge closure problems are, therefore, symptomatic of ill-posed inverse problems. Those researchers using Laurent series expansions and Mellin-Barnes integrals to represent their model flow in the hodograph plane are specifying a pressure distribution around some profile. They must arbitrarily assume a free stream speed to begin their solution process. In so doing, they violate Lighthill's first constraint and closure problems result. Given a closed profile in incompressible flow, the new transformation method will transform it into another closed profile.

The construction of a model flow about a closed lifting profile is only theoretically possible by the method of Chapter 5. A closed lifting profile can result only when the number of vortices is infinite. Thus, in practice a small closure problem will exist over some small interval at the trailing edge and/or the leading edge. By specifying the coordinate points at the nose and trailing edge of the given profile arbitrarily close together, the closure problem is forced to fall over some arbitrarily small interval. Thus, the closure problem can be reduced to effects which may be ignored for engineering purposes.

Another possible drawback to the construction of the model flow by the method presented in Chapter 5 lies with the initial guess of the model flow. The stability of the numerical method is strongly dependent upon this initial guess. However, if the procedure outlined in Chapter 5 is followed, a reasonably accurate initial guess may be constructed which should eliminate the stability problem.

The limiting maximum local Mach number of 2.2735 may be a theoretical drawback, but not a practical one. The highest local Mach number existing in shock-free flow about any airfoil this author has seen is 1.42. This Mach number existed over two airfoils designed by Boerstoel[12].

Chapter 7

CONCLUSION

M. James Lighthill in 1947 developed a method of solution of the hodograph equation governing the subsonic flow around a body. Employing the incompressible flow around a body as a boundary condition, this method transforms the boundary condition into a solution of the hodograph equation. Preventing such a transformation in supersonic flow are the complicated and physically insensible supersonic asymptotic formulae of the Chaplygin functions which Lighthill developed.

This study developed a physicallyreasonable asymptotic formula of the Chaplygin functions valid for supersonic flow up to Mach 2.2735 in air. The new formula permits the development of a method which transforms an incompressible boundary condition into a solution of the hodograph equation governing both subsonic and supersonic flow about a body. Any incompressible flow around a profile that is governed by an incompressible complex potential may be transformed into a solution representing the compressible, transonic, two-dimensional, potential flow about a similar profile. This transformation is developed from, and is markedly similar to, Lighthill's transformation.

Only numerical data concerning the flow velocity magnitude and direction around a given profile are required for the transformation. Numerical methods required by this method are simple and well behaved.

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Appendix A

DERIVATION OF THE GOVERNING EQUATIONS

Any study in fluid dynamics must begin with the governing equation(s) of motion. The governing equation is often derived under certain assumptions which limit its use. These assumptions must be known so that the governing equation(s) may not be incorrectly applied to certain flows.

The following derivations parallel those of Schapiro[1].

Beginning with the assumption of irrotational motion it is easily shown that the integral

$$\int_A^B q \cos\alpha d\ell \tag{A.1}$$

is independent of path. Thus, $q \cos\alpha d\ell$ is an exact differential and one may set:

$$d\phi = q \cos\alpha d\ell \tag{A.2}$$

where ϕ is the velocity potential. Potential flow and irrotational flow are often used interchangeably. Thus, for two-dimensional flow

$$\phi_x \equiv \partial\phi/\partial x = u \tag{A.3}$$

$$\phi_y \equiv \partial\phi/\partial y = v \tag{A.4}$$

where u and v are the velocity components in the x- and y-directions, respectively, of a Cartesian coordinate system. The above is summarized as:

$$q = \nabla \phi \tag{A.5}$$

For a steady two-dimensional flow, the continuity equation becomes

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \ . \tag{A.6}$$

Carrying out the indicated differentions yields

$$\rho(\phi_{xx}+\phi_{yy}) + \phi_x \partial\rho/\partial x + \phi_y \partial\rho/\partial y = 0 \ . \tag{A.7}$$

Next, an equation is required relating ρ with the potential function. Summing Euler's equations of motion along a streamline in steady flow yields:

$$dp = -\rho d(q^2/2) \ . \tag{A.8}$$

Expanding:

$$dp = -\rho d\left(\frac{u^2+v^2}{2}\right) = -\rho d\left(\frac{\phi_x^2+\phi_y^2}{2}\right) \ . \tag{A.9}$$

For isentropic flow, the sound velocity is

$$c^2 = dp/d\rho\,|_s \ . \tag{A.10}$$

Substituting into the above

$$\frac{\partial\rho}{\partial x} = \frac{\partial}{\partial x}\left(\frac{dp}{c^2}\right) = \frac{1}{c^2}\frac{\partial}{\partial x}(dp) = -\frac{\rho}{c^2}(\phi_x\phi_{xx}+\phi_y\phi_{yx}) \ . \tag{A.11}$$

Likewise:

$$\partial\rho/\partial y = -\rho/c^2(\phi_x\phi_{xy}+\phi_y\phi_{yy}) \ . \tag{A.12}$$

Substitution of the above into equation (A.7) yields:

$$\left[1-\frac{\phi_x^2}{c^2}\right]\phi_{xx} + \left[1-\frac{\phi_y^2}{c^2}\right]\phi_{yy} - 2\frac{\phi_x\phi_y}{c^2}\phi_{xy} = 0 \ . \tag{A.13}$$

The local sonic speed is derived from the energy equation for an isentropic flow of an ideal gas. It is:

$$c^2 = c_o^2 - \frac{\gamma-1}{2} q^2 = c_o^2 - \frac{\gamma-1}{2} (\phi_x^2 - \phi_y^2) \quad . \tag{A.14}$$

Note that the isentrope for an ideal gas is given by

$$p/\rho^{\gamma} = \text{constant} \tag{A.15}$$

from which one can see that specifying a straight line isentrope in the p vs. $1/\rho$ plane amounts to specifying $\gamma = -1$.

Equation (3) is the governing equation of the potential function. Obviously non-linear, its solution is at best difficult unless simplifications are made or it is somehow transformed to a linear equation.

The stream function is now defined. The stream function, ψ , exists only for the two-dimensional steady flows and is defined by:

$$\phi_x = u \equiv \rho_o/\rho\psi_y \ ; \quad \phi_y = v \equiv -\rho_o/\rho\psi_x \ . \tag{A.16}$$

Substituting the above into equations (A.13) and (A.14) yields the governing partial differential equation of the stream function:

$$[1-(\frac{\rho_o}{\rho})^2 \frac{\psi_y^2}{c^2}] \psi_{xx} + [1-(\frac{\rho_o}{\rho})^2 \frac{\psi_x^2}{c^2}] \psi_{yy} + 2(\frac{\rho_o}{\rho})^2 \frac{\psi_x \psi_y}{c^2} \psi_{xy} = 0 \tag{A.17}$$

$$c^2 = c_o^2 - \frac{\gamma-1}{2} (\frac{\rho_o}{\rho})^2 (\psi_x^2 + \psi_y^2) \ . \tag{A.18}$$

The conservation of energy for the isentropic flow of an ideal gas is

$$q^2 + 2C_pT = \text{constant} \ . \tag{A.19}$$

For an ideal gas

$$C_pT = (C_p/\gamma R)\ \gamma RT = (1/\gamma-1)c^2\ . \tag{A.20}$$

Combining these two equations and evaluating the constant at three reference conditions (zero speed, zero temperature, and sonic speed) yields:

$$q^2 + [2/\gamma-1]\,c^2 = [2/\gamma-1]\,c_o^{\,2} \tag{A.21}$$

$$= q_m^{\,2} \tag{A.22}$$

$$= (\gamma+1/\gamma-1)\ c^2 = (\gamma+1/\gamma-1)\ q^2\ . \tag{A.23}$$

Equations (A.21) and (A.22) may be recast as

$$\frac{\gamma}{\gamma-1}\ \frac{p}{\rho} + \frac{q^2}{2} = \frac{\gamma}{\gamma-1}\ \frac{p_o}{\rho_o} \tag{A.24}$$

$$(2\gamma/\gamma-1)\ p_o/\rho_o = q_m^{\,2} \tag{A.25}$$

by noting $c^2 = \gamma p/\rho$ for isentropic flow. Solving equation (A.24) for q and employing the isentrope relation yields the De Saint Venant and Wantzel formula:

$$q^2 = q_m^2[1 - (\frac{\rho}{\rho_o})^{\gamma-1}]\ . \tag{A.26}$$

The final required relation is found by substituting the above into equation (A.21) yielding:

$$\frac{q^2}{2} + \frac{c^2}{\gamma-1} = \frac{q^2}{2} + \frac{q_m^2}{2}\ (\frac{\rho_o}{\rho})^{\gamma-1}\ . \tag{A.27}$$

Appendix B

The Molenbroek-Chaplygin Transformation

In equations (A.13) and (A.17) the independent variables are the Cartesian coordinates (x,y). By considering these coordinates as functions of the polar velocity coordinates (q,θ) the governing equations may be linearized. This is the Molenbroek-Chaplygin transformation.

The following derivations follow those of Boerstoel[5] and Ferrari and Tricomi[2].

It follows from the definitions of the stream and potential functions that their total derivations may be related thusly:

$$d\phi+i(\rho_o/\rho)d\psi=udx+vdy+i(udy-vdx)$$

$$=(u-iv)(dx+idy)=qe^{-i\theta}dz \qquad (B.1)$$

Thus:

$$dz=e^{i\theta}/q[d\phi+i(\rho_o\rho)d\psi] \qquad (B.2)$$

$$\partial z/\partial q=e^{i\theta}/q[\phi_q+i(\rho_o/\rho)\psi_q] \qquad (B.3)$$

$$\partial z/\partial\theta=e^{i\theta}/q[\phi_\theta+i(\rho_o/\rho)\psi_\theta] \qquad (B.4)$$

When x and y are considered as continuously differentiable functions of q and θ, their mixed partial derivatives exist and the order or differentiation is immaterial. Hence

$$\{e^{i\theta}/q[\phi_q+i(\rho_o/\rho)\psi_q]\}_\theta$$

$$=\{e^{i\theta}[\phi_\theta+i(\rho_o/\rho)\psi_\theta]\}_q \qquad (B.5)$$

yielding:

$$ie^{i\theta}/q[\phi_q+i(\rho_o/\rho)\psi_q]$$

$$= -e^{i\theta}/q^2\ \phi_\theta+ie^{i\theta}(\rho_o/\rho q)_q\psi_\theta\ . \qquad (B.6)$$

Equating the real and imaginary parts yields:

$$\phi_\theta=q(\rho_o/\rho)\psi_q \qquad (B.7)$$

$$\phi_q=q(\rho_o/\rho q)_q\psi_\theta\ . \qquad (B.8)$$

We define

$$\tau=(q/q_m)^2 \qquad (B.9)$$

Subtracting equation (A.20) from equation (A.26) yields:

$$\rho_o/\rho=(1-\tau)^{-1/\gamma-1}\ . \qquad (B.10)$$

Equations (B.7) and (B.8) can now be expressed in terms of τ .

Noting

$$d\tau/dq=2q/q_m^{\ 2} \qquad (B.11)$$

we obtain:

$$\phi_\theta=2(q/q_m)^2(\rho_o/\rho)\psi_\tau \qquad (B.12)$$

$$\phi_\theta=2\tau(1-\tau)^{-1/\gamma-1}\psi_\tau \qquad (B.13)$$

$$\phi_\theta=Q\psi_\tau \qquad (B.14)$$

$$Q=2\tau(1-\tau)^{-1/\gamma-1}\ . \qquad (B.15)$$

Likewise:

$$\phi_\tau=q(\rho_o/\rho q)_\tau\psi_\theta \qquad (B.16)$$

$$\frac{\partial}{\partial\tau}\left(\frac{\rho_o}{\rho q}\right)=\left(\frac{\rho_o}{q}\right)\frac{d}{d\tau}\left(\frac{1}{\rho}\right)+\left(\frac{\rho_o}{\rho}\right)\frac{d}{d\tau}\left(\frac{1}{q}\right) \qquad (B.17)$$

$$q\ \partial/\partial\tau(\rho_o/\rho q)=(\gamma-1)^{-1}(1-\tau)^{-\gamma/\gamma-1}-1/2\tau(1-\tau)^{-1/\gamma-1} \qquad (B.18)$$

$$= -1/2\tau(1-\tau)^{\gamma/\gamma-1}[1-\tau-(2\tau/\gamma-1)] \tag{B.19}$$

$$= -1/2\tau(1-\tau)^{\gamma/\gamma-1}[1-\tau(\gamma+1/\gamma-1)] \ . \tag{B.20}$$

From equation (A.20)

$$q^2/2 + c^2/\gamma-1 = q_m{}^2/2$$

we see when $q = c$

$$\tau_s = (c/q_m)^2 = \gamma-1/\gamma+1 \ . \tag{B.21}$$

Hence:

$$\psi_\theta = P\phi_\tau \tag{B.22}$$

$$P= -2\tau(1-\tau)^{\gamma/\gamma-1}/(1-\tau/\tau_s) \tag{B.23}$$

Cross-differentiating equations (B.14) and B.22) to eliminate the potential functions yields:

$$PQ\psi_{\tau\tau}+PQ_\tau\psi_\tau-\psi_{\theta\theta}=0 \tag{B.24}$$

This is the linear, governing mixed partial differential equation of motion in the hodograph plane in terms of the stream function. The discriminant of the above is:

$$B^2-4AC=0-4(PQ)(-1)=4PQ \tag{B.25}$$

The coefficient PQ is negative when $\tau<\tau_s$. Thus, equation (B.24) is elliptic for subsonic flow. Likewise, PQ is positive when $\tau>\tau_s$ and equation (B.24) is hyperbolic for supersonic flow. The mixed behavior is evident.

The variables τ and θ are separated by

$$\psi(\tau,\theta)=\psi_n(\tau)e^{\pm in\theta}$$

yielding:

$$PQ\psi_{n\tau\tau}+PQ_\tau\psi_{n\tau}+n^2\psi_n=0 \ . \tag{B.26}$$

Substituting $\psi_n(\tau)=\tau^{n/2}F_n(\tau)$ into equation (B.26) yields:

$$PQF_n'' + [PQ(n/\tau 2) + PQ_\tau]F_n'$$
$$+[PQ(n/2)(n/2-1)\tau^{-2} + PQ(n/2\tau) + n^2]F_n = 0 \tag{B.27}$$

where the prime denotes differentiation with respect to τ . Substituting:

$$PQ = -4\tau^2(1-\tau)/(1-\tau/\tau_s) \; ;$$
$$PQ_\tau = -4\tau[1+(2-\gamma)/(\gamma-1)\tau]/(1-\tau/\tau_s) \tag{B.28}$$

into equation (B.27), collecting and cancelling terms, yields:

$$\tau(1-\tau)F_n''(\tau) + [n+1-(n-(2-\gamma)/(\gamma-1)\tau]F_n'(\tau)$$
$$+n(n+1)/2(\gamma-1)\; F_n(\tau) = 0 \tag{B.29}$$

Comparison with the hypergeometric equation of which the Gaussian hypergeometric functions are solutions shows $F_n(\tau)$ is a Gaussian hypergeometric function. In this case:

$$a_n b_n = -n(n+1)/2(\gamma-1) \tag{B.30}$$

$$a_n + b_n = n-(1/\gamma-1) \tag{B.31}$$

$$c_n = n+1 \; . \tag{B.32}$$

Solving for a_n, b_n yields:

$$a_n, b_n = 1/2[n-1/\gamma-1 \pm ((\gamma+1)n^2/\gamma-1 - (1/\gamma-1)^2)^{1/2}] \tag{B.33}$$

where, by convention, $a_n > b_n$. The Gaussian hypergeometric functions, found by the Method of Frobenius about $\tau = 0$ (other regular singular points are 1, ∞), are:

$$F(a,b;c;\tau) = 1+(ab/c)\tau + a(a+1)b(b+1)/2!c(c+1)\tau^2$$
$$+(a(a+1)(a+2)(b)(b+1)(b+2)/3!c(c+1)(c+2)\tau^3 \tag{B.34}$$

The Chaplygin functions, $\psi_n(\tau)$, are defined as:

$$\psi_n(\tau) = \tau^{n/2}F(a_n, b_n; n+1; \tau) \tag{B.35}$$

Correspondence between the hodograph and physical planes is one-to-one when the Jacobian of the transformation is not zero or infinite.

$$J = \partial(x,y)/\partial(q,\theta)$$

$$= (\partial)/\partial q)(\partial y/\partial\theta)-(\partial x/\partial\theta)(\partial y/\partial q) \qquad (B.36)$$

From equations (B.3) and (B.4), and noting $Z=x+iy$, the Jacobian is seen to be

$$J = e^{2i\theta}/q^2 \ (\rho_o/\rho)[\phi_q\psi_\theta-\phi_\theta\psi_q] \ . \qquad (B.37)$$

Substituting equations (B.7) and (B.8) into the above yields:

$$J = e^{2i\theta}/q(\rho_o/\rho)[(\rho_o/\rho q)_q\psi_\theta{}^2-(\rho_o/\rho)\psi_q{}^2] \qquad (B.38)$$

Noting:

$$\partial/\partial q(\rho_o/\rho q) = (\rho_o/\rho)(-q^{-2})+(\rho_o/q)d/dq(1/\rho)$$

$$= -\rho_o/\rho q^2+\rho_o/\rho c^2$$

$$= \rho_o/\rho(1/c^2-1/q^2) \qquad (B.39)$$

equation (B.38) becomes:

$$J = e^{2i\theta}/q(\rho_o/\rho)^2[(1/c^2-1/q^2)\psi_\theta^2-\psi_q^2] \qquad (B.40)$$

$$J = -e^{2i\theta}/q^3(\rho_o/\rho^2)[q^2\psi_q{}^2-(M^2-1)\psi_\theta{}^2] \qquad (B.41)$$

For a one-to-one correspondence between the hodograph and physical planes, we must have:

$$q \neq o; \ q^2\psi_q^2-(M^2-1)\psi_\theta{}^2 \neq o,\infty \qquad (B.42)$$

When the Jacobian is infinite, branch points or branch lines occur. Branch-points exist in subsonic flow and branch lines exist in supersonic flow. An obvious branch point is the stagnation point where $q = o$. Ferrari and Tricomi[2], show that no singularity arises in the fluid flow at branch line images for supersonic flow.

Limit lines occur when the Jacobian is zero. In subsonic flows, the Jacobian could be zero at the stagnation point or at a point of infinite curvature. Supersonic flows, however, can have lines where the Jacobian is zero. The equations defining these lines are:

$$q\psi_q - \sqrt{M^2-1}\ \psi_\theta = 0 \tag{B.43}$$

$$q\psi_q + \sqrt{M^2-1}\ \psi_\theta = 0 \tag{B.44}$$

A theorem due to von Karman states that in the hodograph plane the limit lines are the loci of points of tangency of the stream lines with the characteristics of the governing equation. The slope of a characteristic in this case is:

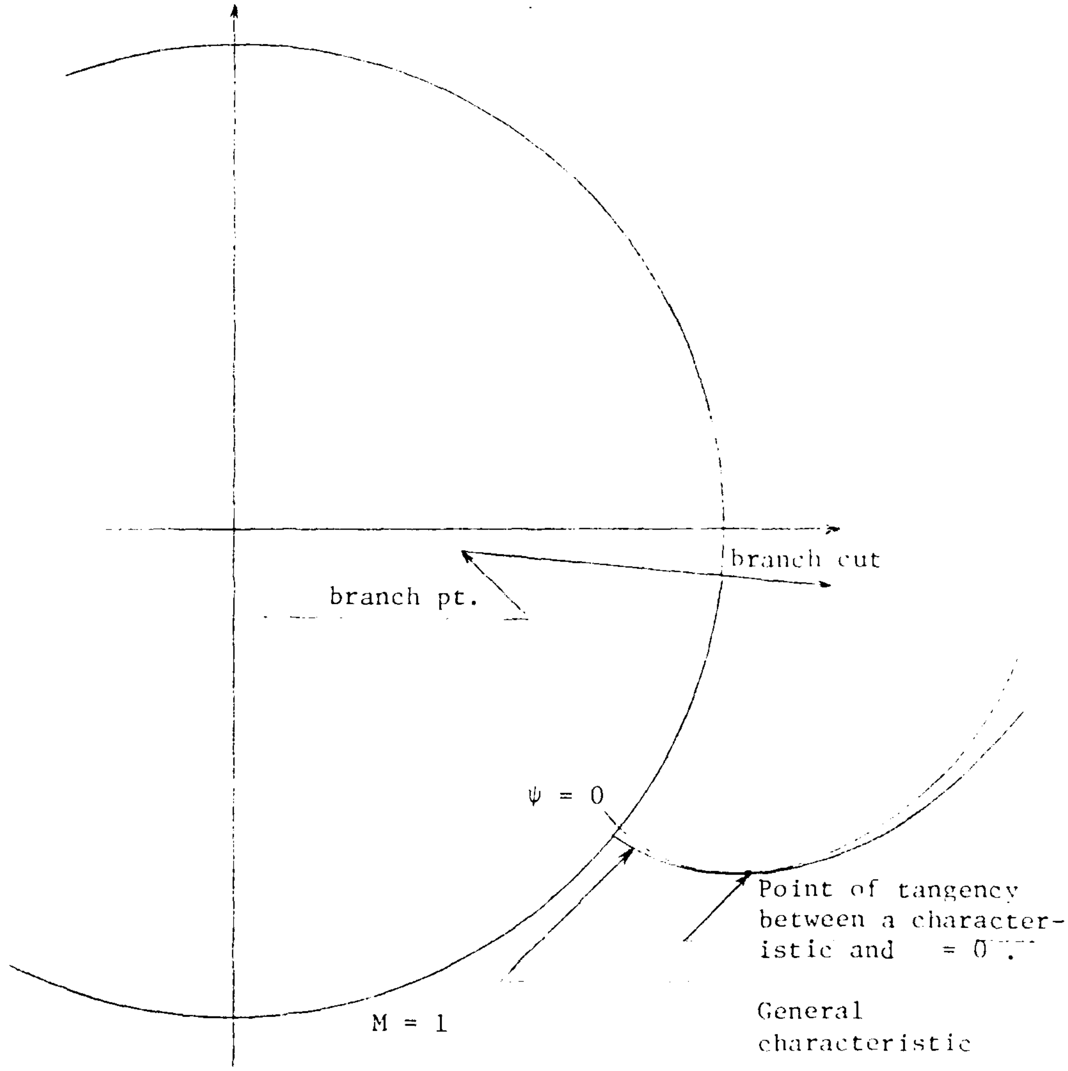
$$d\theta/d\tau = -B\pm\sqrt{B^2-4AC}\ /2A = \pm\ 1/\sqrt{PQ} \tag{B.45}$$

See Figure 2.

A qualitative discussion will help the visualization of limit lines. Because the Jacobian is zero, a finite length in the hodograph plane is mapped to zero length in the physical plane. Thus, the physical plane could be considered to be folded at the image of a limit line. Although the velocity will be a smooth function about the limit line in the hodograph, it will be discontinuous in the physical plane at the limit line image. The presence of a limit line predicts a shock in the physical flow.

FIGURE 2

Appearance of a Limit Line



Appendix C

THE HODOGRAPH PLANE FOR A TYPICAL LIFTING AIRFOIL

The nature of the hodograph plane depicting the flow about a typical lifting airfoil will be illuminated by three examples.

Non-Lifting Circular Cylinder in Incompressible Flow

Consider first the complex potential, $\Phi^{(i)} = \phi^{(i)} + i\psi^{(i)}$, of an incompressible uniform flow about a circle. With the circle radius and free stream velocity normalized, this potential is

$$\Phi^{(i)} = Z_c + Z_c^{-1} \tag{C.1}$$

and

$$d\Phi^{(i)}/dz_c = u - iv = qe^{i\theta} = 1 - Z_c^{2} \tag{C.2}$$

Defining $\zeta_c = qe^{i\theta}$ and substituting into the above yields:

$$Z_c = \pm[(1-\zeta_c)^{-1/2} \tag{C.3}$$

$$\Phi^{(i)} = \pm[(1-\zeta_c)^{1/2} + (1-\zeta_c)^{-1/2}] \; . \tag{C.4}$$

Thus, ζ_c must be defined on a two-sheeted Riemann surface for an unequivocal one-to-one correspondence to exist. Branch points exist when

$$d\zeta_c/dZ_c = 0, \infty \; . \tag{C.5}$$

Performing the differentiation yields

$$d\zeta_c/dZ_c = 2/Z_c^3 \tag{C.6}$$

Possible branch points exist when $Z_c = o$ and when $Z_c = \infty$

This point, not in the external flow, is therefore of no interest. When $Z_c = \infty$, $\zeta_c = 1$ and one is at the force stream conditions. The branch cut extends from $\zeta_c = 1$ to infinity along the real axis. Note also that $\zeta_c = 1$ is a singular point of equations (C.3) and (C.4).

Consider the stream function about the circle. It will obviously be antisymmetric with respect to the axis of flow. Let's look, therefore, at the flow above the axis of symmetry. The front stagnation point maps to the point (o , $\pi/2$) on the hodograph. At the point of greatest cross-section presented to the flow, the flow angle is zero while the flow magnitude is a maximum. Beyond this point, the flow angle becomes negative and the magnitude decreases until at the rear stagnation point the velocity is zero again while the flow angle is $-\pi/2$. See Figure 3.

Lifting Circular Cylinder in Incompressible Flow

Consider next the same situation with circulation. The complex potential for the incompressible flow is now:

$$\phi^{(i)} = Z_c + Z_c^{-1} + i\Gamma/2\pi \; \ell n Z_c \tag{C.7}$$

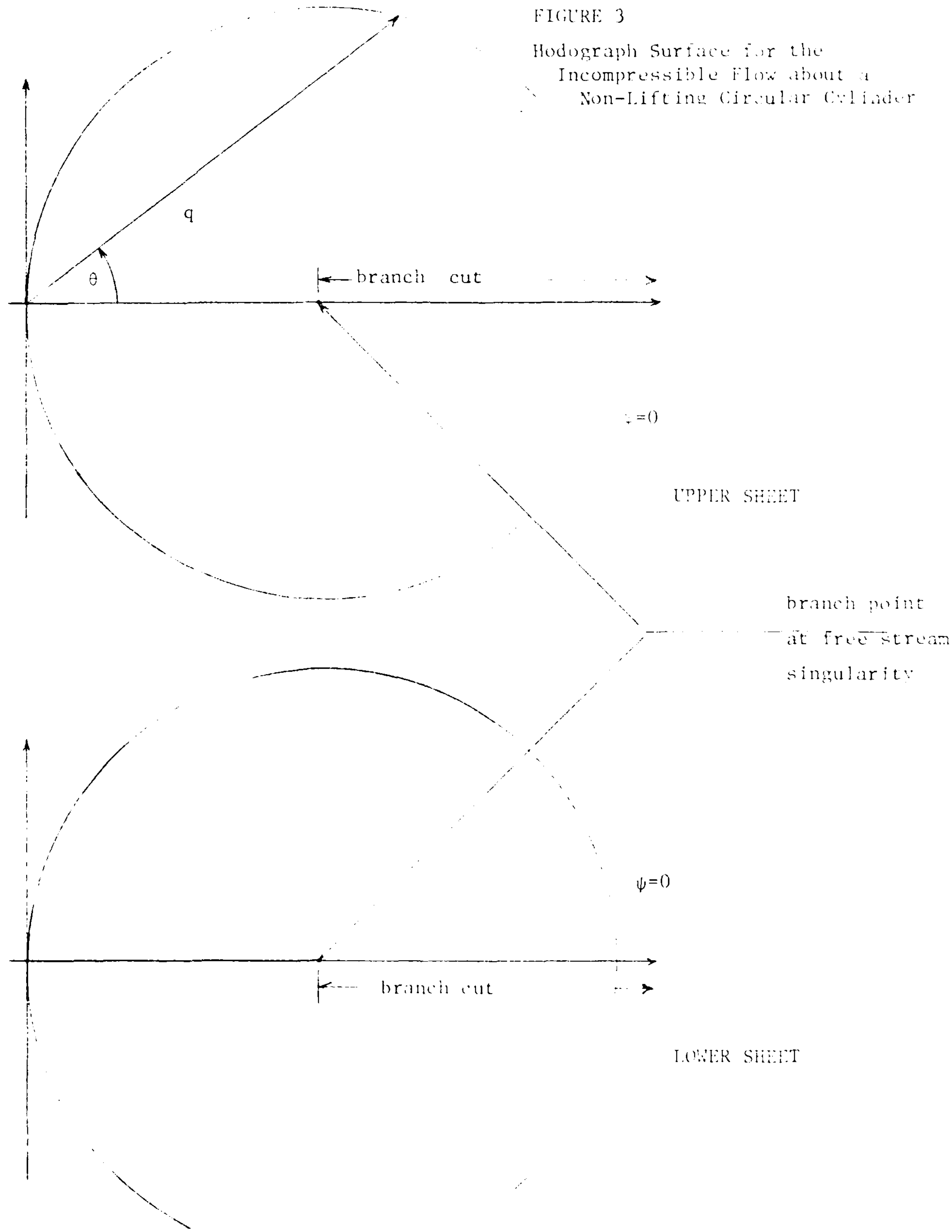
and

$$d\phi^{(i)}/dZ_c = u-iv = \zeta_c = 1-Z_c^{-2} + i\Gamma/2\pi \; Z_c^{-1} \tag{C.8}$$

We see the hodograph plane is again two sheeted. Finding the possible branch points:

FIGURE 3

Hodograph Surface for the Incompressible Flow about a Non-Lifting Circular Cylinder



$$d\zeta_c/dZ_c = 2/Z_c^{\,3} - i\Gamma/2\pi\ Z_c^{-2} \tag{C.9}$$

Thus:

$$d\zeta_c/dZ_c = 0 \quad \text{at} \quad Z_c = \infty \tag{C.10}$$

$$d\zeta_c/dZ_c = 0 \quad \text{at} \quad X_c = -4\pi i/\Gamma \tag{C.11}$$

$$d\zeta_c/dZ_c = \infty \ \text{at} \quad Z_c = o \tag{C.12}$$

The corresponding points in the hodograph plane are, respectively:

$$\zeta_c = 1 \tag{C.13}$$

$$\zeta_c = 1 - (\Gamma/4\pi)^2 \tag{C.14}$$

$$\zeta_c = \infty \tag{C.15}$$

The point $Z_c = o$ is again of no practical interest. From equation (C.8), one sees that $\zeta_c = 1$ can be reached from two values of Z_c, namely:

$$Z_c = \infty \tag{C.16}$$

$$Z_c = -2\pi i/\Gamma \tag{C.17}$$

Therefore, the point $\zeta_c = 1$ cannot be the branch point. Indeed, the value of ζ_c corresponding to the free stream conditions ($Z_c = \infty$) is a singular point again and must not be defined on the sheet containing the regular point $Z_c = -2\pi i/\Gamma$. One is left with

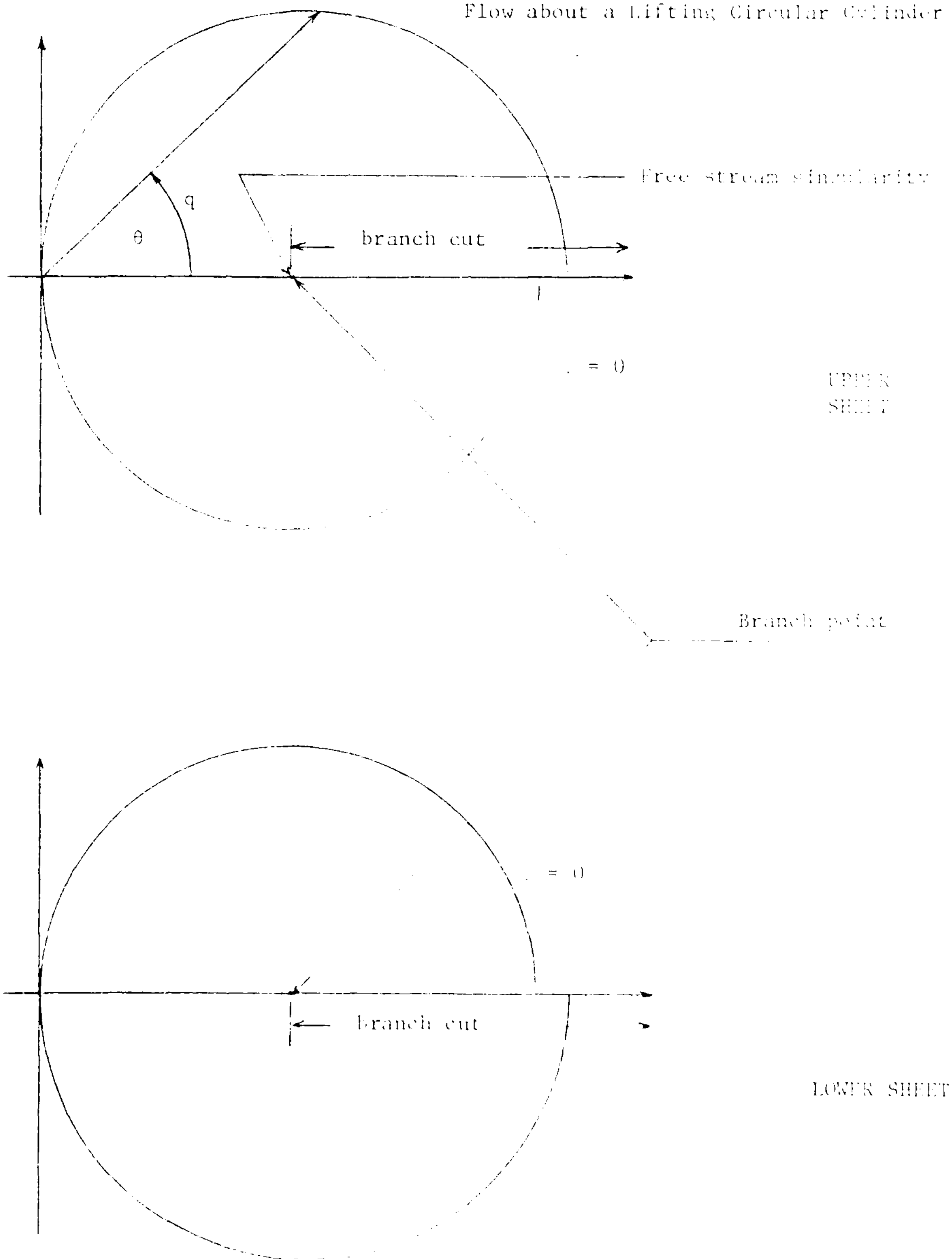
$$\zeta_c^* = 1 - (\Gamma/4\pi)^2 \tag{C.18}$$

as the last possibility. Substituting this into equation (C.8) shows that ζ_c^* can be reached from one and only one point in the physical plane. Thus, ζ_c^* is the branch point and the branch cut runs along the real axis out to infinity.

Qualitatively, the physical flow image in the hodograph

FIGURE 4

Hodograph Surface for the Incompressible Flow about a Lifting Circular Cylinder



plane is similar to the non-circulatory case. Here, however, th stream function is not antisymmetric with respect to the flow axis a the maximum velocity magnitude over the top half of the cylinder i greater than that obtained over the lower half. See Figure 4.

Lifting Airfoil in Compressible Flow

Consider a lifting airfoil. The incompressible comple potential may be conformally mapped from an incompressible flo representing that about a circle. This general mapping i (Glauert[13]):

$$Z = Z_c + a_1 Z_c^{-1} + a_2 Z_c^{-2} + \ldots \qquad \text{(C.19}$$

where the coefficients, a_i , are generally complex. Now:

$$\zeta = d\phi^{(i)}/dz = (d\phi^{(i)}/dZ_c)(dZ_c/dZ) = \zeta_c(dZ/dZ_c)^{-1} \qquad \text{(C.20}$$

$$dZ/dZ_c = 1 - a_1 Z_c^{-2} - 2a_2 Z_c^{-3} \ldots \qquad \text{(C.21}$$

Assume Z_c larg enough to neglect orders greater than Z_c^{-2} . Then:

$$(dZ/dZ_c)^{-1} = 1 + a_1 Z_c^{-2} + a_1^2 Z_c^{-4} + \ldots \qquad \text{(C.22}$$

Equation (C.20) becomes:

$$\zeta = (1 - Z_c^{-2} + i\Gamma/2\pi\ Z_c^{-1})(1 + a_1 Z_c^{-2} + a_1^2 Z_c^{-4} + \ldots) \qquad \text{(C.23}$$

$$\zeta = 1 + i\Gamma/2\pi\ Z_c^{-1} - (1 - a_1) Z_c^{-2} + O(Z_c^{-3}) \qquad \text{(C.24}$$

Comparing equations (C.8) and (C.24) shows the lifting airfoil i similar to the lifting circular cylinder except that the branch poin has moved off the real axis and the branch cut is now a radial ra beginning at the branch point

$$\zeta^* = 1 - \Gamma^2/16\pi^2(1 - a_1)$$

... extending to infinity. The point corresponding to the free strea conditions, $\zeta = 1$, is still a singular point.

The hodograph plane still has the same basic structure in compressible flow as it did in incompressible flow. For low speeds one may use small perturbation theory to linearize the governing equation, (A.13). The result is:

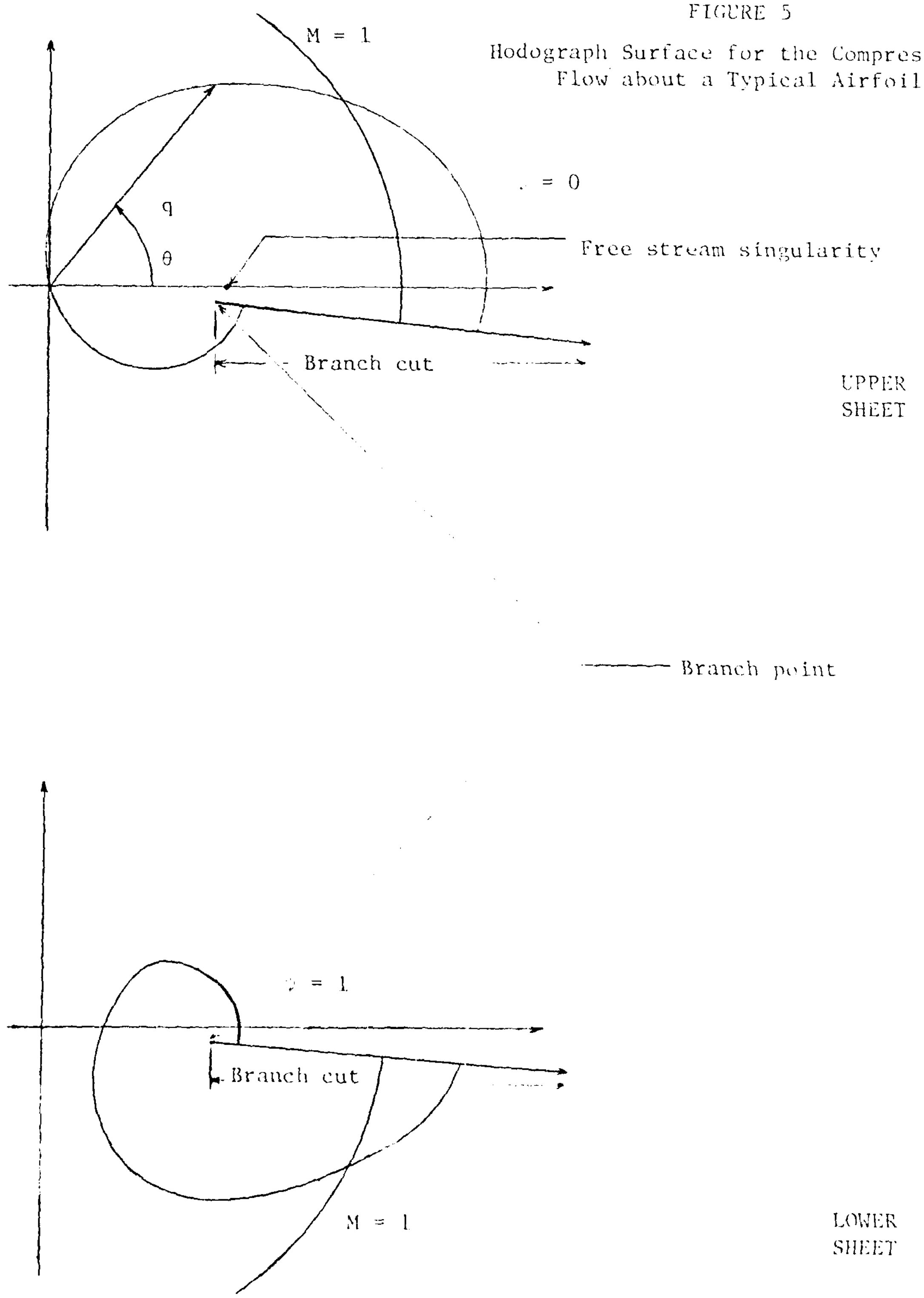
$$\beta_\infty^{2} \quad \partial^2\phi/dx^2 + \partial^2/\phi/\partial y^2 = 0$$

$$\beta_\infty = \sqrt{1-M_\infty^{2}}$$

Through Gothert's Rule and the Prandtl-Glauert Similarity Rules, one can predict that the compressible complex potential is the same as the incompressible case for affinely related airfoils. However, the previous result is for a perfectly general airfoil in incompressible flow. Thus, the same result will hold for any general airfoil affinely related to this general airfoil. In particular, the point $\zeta = 1$ is still a singular point and the hodograph is still two-sheeted. See Figure 5.

FIGURE 5

Hodograph Surface for the Compressible Flow about a Typical Airfoil



Appendix D

INTRODUCTORY RELATIONS REQUIRED IN ASYMPTOTIC FORMULAE DERIVATIONS

Recall the characteristic equations of equation (1.6):

$$d\theta/d\tau = \pm (PQ)^{-1/2} \qquad \text{(D.1)}$$

Consider first the case where $d\theta/d\tau$ is real, i.e. supersonic flow. Then,

$$d\theta/d\tau = \pm 1/2\tau((\tau/\tau_s - 1)/(1-\tau))^{1/2} \qquad \text{(D.2)}$$

$$d\theta/d\tau = \pm 1/2\tau([(\gamma+1)\tau-(\gamma-1)]/(1-\tau)(\gamma-1))^{1/2} . \qquad \text{(D.3)}$$

Hence:

$$\int^{t} d\theta = \pm\int_{\gamma-1/\gamma+1}^{\tau}((\gamma+1)\tau-(\gamma-1)/(1-\tau)(\gamma-1))^{1/2}\, d\tau/2\tau \qquad \text{(D.4)}$$

$$t = \pm\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{(\gamma+1)\tau-(\gamma-1)}{(1-\tau)(\gamma+1)}} - \tan^{-1}\sqrt{\frac{(\gamma+1)\ \ -(\gamma-1)}{(1-\tau)(\gamma-1)}} \qquad \text{(D.5)}$$

See Figure 6 for a plot of t versus τ. The governing equation (1.6) for strictly supersonic flow becomes

$$PQ[\psi_t t_{\tau\tau} + \psi_{\tau\tau} t_\tau^2] + PQ_\tau \psi_t t_\tau - \psi_{\theta\theta} = 0 \qquad \text{(D.6)}$$

$$t_\tau^2 = (PQ)^{-1} \qquad \text{(D.7)}$$

$$\psi_{tt} + (PQt_{\tau\tau} + PQ_\tau t_\tau)\psi_t - \psi_{\theta\theta} = 0 \qquad \text{(D.8)}$$

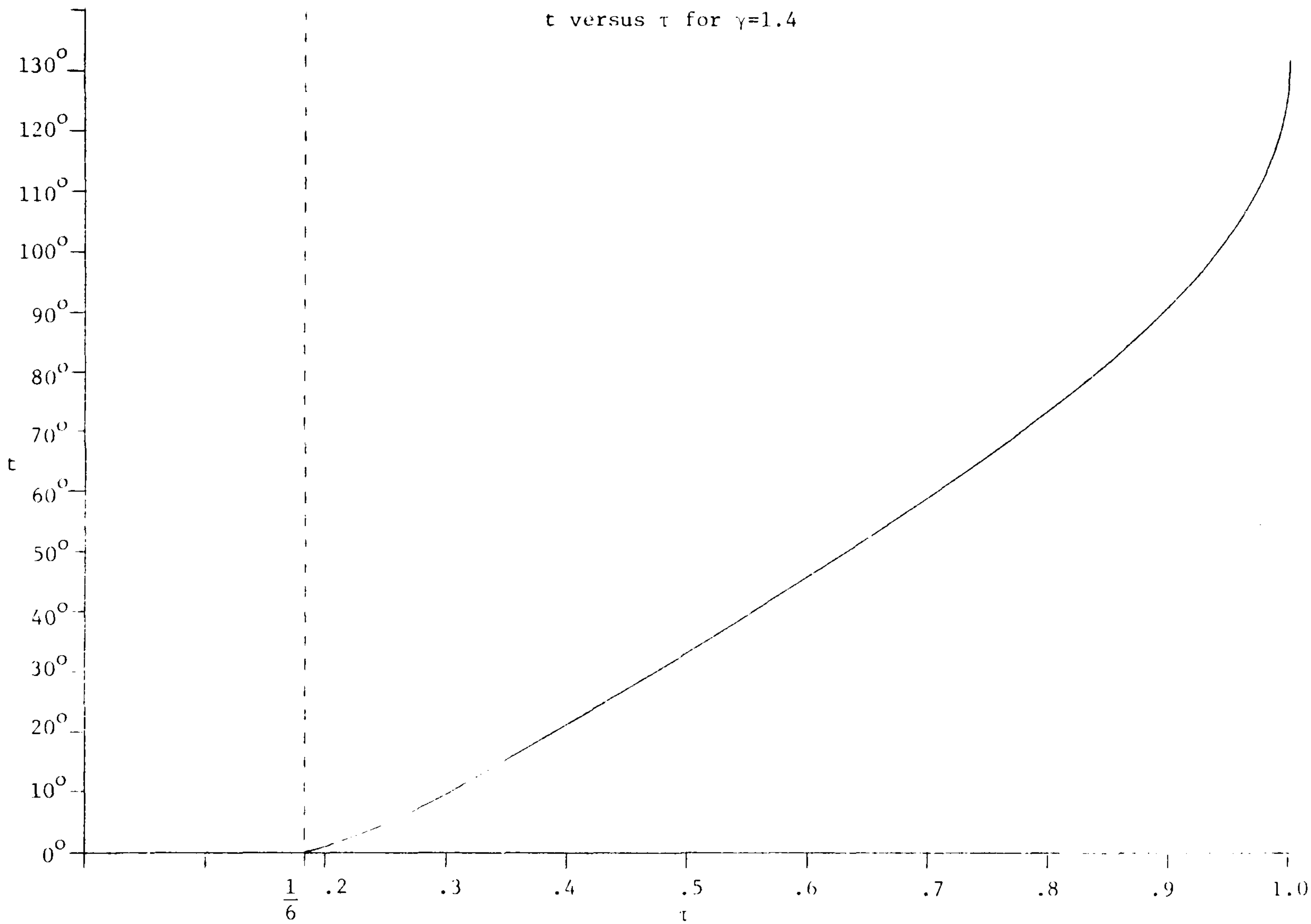
or

$$\psi_{tt} + S(t)\psi_t - \psi_{\theta\theta} = 0 \qquad \text{(D.9)}$$

$$S(t) = PQ[\mp (PQ)^{-3/2}(P_\tau Q + Q_\tau)] + PQ_\tau(\pm(PQ)^{-1/2}) \qquad \text{(D.10)}$$

FIGURE 6

t versus τ for $\gamma=1.4$



$$S(t)=\pm 1/2(PQ)^{-1/2}(PQ_{\tau}-P_{\tau}Q) \qquad \text{(D.11)}$$

$$S(t)=\pm 1/2(Q_{\tau}/Q-P_{\tau}/P)$$

$$=\pm 2(\gamma+1)/(\gamma-1)^{2}\tau^{2}(1-\tau)^{-1/2}((\gamma+1/\gamma-1)\tau-1)^{-3/2} \qquad \text{(D.12)}$$

In subsonic flow, $P < 0$ and the characteristics are imaginary. One may write:

$$i\, d\theta/d\tau = \pm i(PQ)^{-1/2} = \pm(-PQ)^{-1/2} . \qquad \text{(D.13)}$$

Because the characteristics are imaginary and do not physically exist, Lighthill[3] is valid in redefining the characteristic equation as

$$d\theta/d\tau = (-PQ)^{-1/2} . \qquad \text{(D.14)}$$

Thus:

$$\int_{0}^{s} d\theta = i\int_{0}^{\tau}(((\gamma+1)\tau-(\gamma+1))/(1-\tau)(\gamma-1))^{1/2}\, d\tau/2\tau . \qquad \text{(D.15)}$$

Integrating,

$$s = \sigma+\sqrt{\frac{\gamma+1}{\gamma-1}}\,\tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma+1)(1-\tau)}} \;-\tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}} \qquad \text{(D.16)}$$

where σ is an arbitrary constant yet to be determined. For strictly subsonic flow, the governing equation (1.6) becomes:

$$\psi_{ss} + \psi_{\theta\theta} = T(s)\psi_{s} \qquad \text{(D.17)}$$

$$T(s)=-1/2(-PQ)^{1/2}[Q_{\tau}/Q-P_{\tau}/P]$$

$$=2(\gamma+1)/(\gamma-1)^{2}\;\tau^{2}(1-\tau)^{-1/2}(1-(\gamma+1/\gamma-1)\tau)^{-3/2} \qquad \text{(D.18)}$$

Lighthill[3] defines σ to be that value which causes τ to be asymptotically e^{2s} as $\tau \longrightarrow 0$ and $S \longrightarrow -\infty$. Symbolically, this is represented as $\tau \sim e^{2s}$.

This exponential form is particularly well suited to a series solution of equation (D.17) which will yield the important asymptotic formula for the Chaplygin functions for subsonic flow. The point $\tau = o$, $s = -\infty$, at which $\tau \sim e^{2s}$, represents the incompressible limit. The behavior of any solution to compressible flow - which will include $\psi_n(\tau)$'s - must reduce to the solutions of the Laplace equation in the incompressible limit. This requires thorough knowledge of the behavior of τ as $\tau \rightarrow o$ becomes $\psi_n(\tau)$ is a function τ . This matching of the compressible solutions to the incompressible solutions is the basis of Lighthill's transformation method. Thus,

$$\tau = \mathrm{EXP}\left\{2\left(\sigma + \sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma+1)(1-\tau)}} - \tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}}\right)\right\} . \tag{D.19}$$

By the identity $\tanh^{-1}(x) = 1/2\ \ell n|1+x/1-x|$, $x^2 < 1$, we have:

$$\tau = (e^{2\sigma})\left(\frac{1-x}{1+x}\right)\left(\frac{1+x'}{1-x'}\right)^{\sqrt{\frac{\gamma+1}{\gamma-1}}} \tag{D.20}$$

$$x \equiv \sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma+1)(1-\tau)}} \quad ; \quad x' \equiv \sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}} . \tag{D.21}$$

Thus,

$$2\sigma = \ell n|\tau| + \ell n\left|\frac{1+x}{1-x}\right| - \sqrt{\frac{\gamma+1}{\gamma-1}}\,\ell n\left|\frac{1+x'}{1-x'}\right| \tag{D.22}$$

$$\sigma = \frac{1}{2}\ell n|\tau| - \sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma+1)(1-\tau)}} + \tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}} . \tag{D.23}$$

As $\tau \rightarrow o$,

$$\sigma = -\sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}} + \frac{1}{2}\ell n|\tau| + \tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}} \tag{D.24}$$

The second and third terms approach $-\infty$ and $+\infty$, respectively. Using the binomial expansion and neglecting terms of second order and higher in τ , one deduces:

$$\frac{1}{2}\ell n|\tau| + \tanh^{-1}\sqrt{\frac{(\gamma-1)-(\gamma+1)}{(\gamma-1)(1-\tau)}} = \frac{1}{2}\ell n\left|\frac{2+\frac{\tau}{2}\left(1-\frac{\gamma+1}{\gamma-1}\right)^{-1}}{(\gamma-1)}\right| \qquad (D.25)$$

Neglecting the τ term yields:

$$1/2\ \ell n|2(\gamma-1)|\ . \qquad (D.26)$$

Finally:

$$\sigma = 1/2\ell n|2(\gamma-1)| - \sqrt{\gamma+1/\gamma-1}\ \tanh^{-1}\sqrt{\gamma-1/\gamma+1}\ . \qquad (D.27)$$

For air, $\gamma = 1.4$, $\sigma = -1.173$. See Figure 7 for the plot of s versus τ . Note that $s \longrightarrow \sigma$ as $\tau \longrightarrow \tau_s$. See Figure 8 for the plots of $T(s)$ and $+S(t)$ versus τ .

Theorem 1 (Lighthill[3]) If $o < \tau \leq 1$, $\psi_n(\tau)$ is an analytic function of n except at $n = -2,-3,-4,\ldots$, where it has simple poles, its residue at $n = -m$ being $-mC_m\psi_m(\tau)$, where C_m is positive and $\sim(2\pi m)^{-1}e^{-2\sigma m}$ as $m \rightarrow \infty$.

For any m ,

$$C_m = \Gamma(a_m)\Gamma(1+m-b_m)/\Gamma(a_m-m)\Gamma(1-b_m)(m!)^2 \qquad (D.28)$$

where the values of a_m , b_m are those values previously defined for the hypergeometric function, equation (B.33) .

Lighthill[3] also deduced:

$$\psi_o = 1 \qquad (D.29)$$

$$\psi_1 = (\gamma-1)\{1-(1-\tau)^{\gamma/\gamma-1}\}/\gamma\sqrt{\tau} \qquad (D.30)$$

$$\psi_1 = \tau^{-1/2} + (2\gamma-2)^{-1}\psi_1(\tau) \qquad (D.31)$$

Theorem 2 (Lighthill[3]) If $\delta > o$ and $\sigma_1 < \sigma$, then $W_n(s) \longrightarrow 1$, i.e. $\psi_n(\tau) \sim e^{ns}V(\tau)$, uniformly for $s \leq \sigma_1$, and for n in the whole complex plane with circles of radius δ round each negative integer excluded as $|n| \longrightarrow \infty$.

FIGURE 7

s versus τ for $\gamma=1.4$

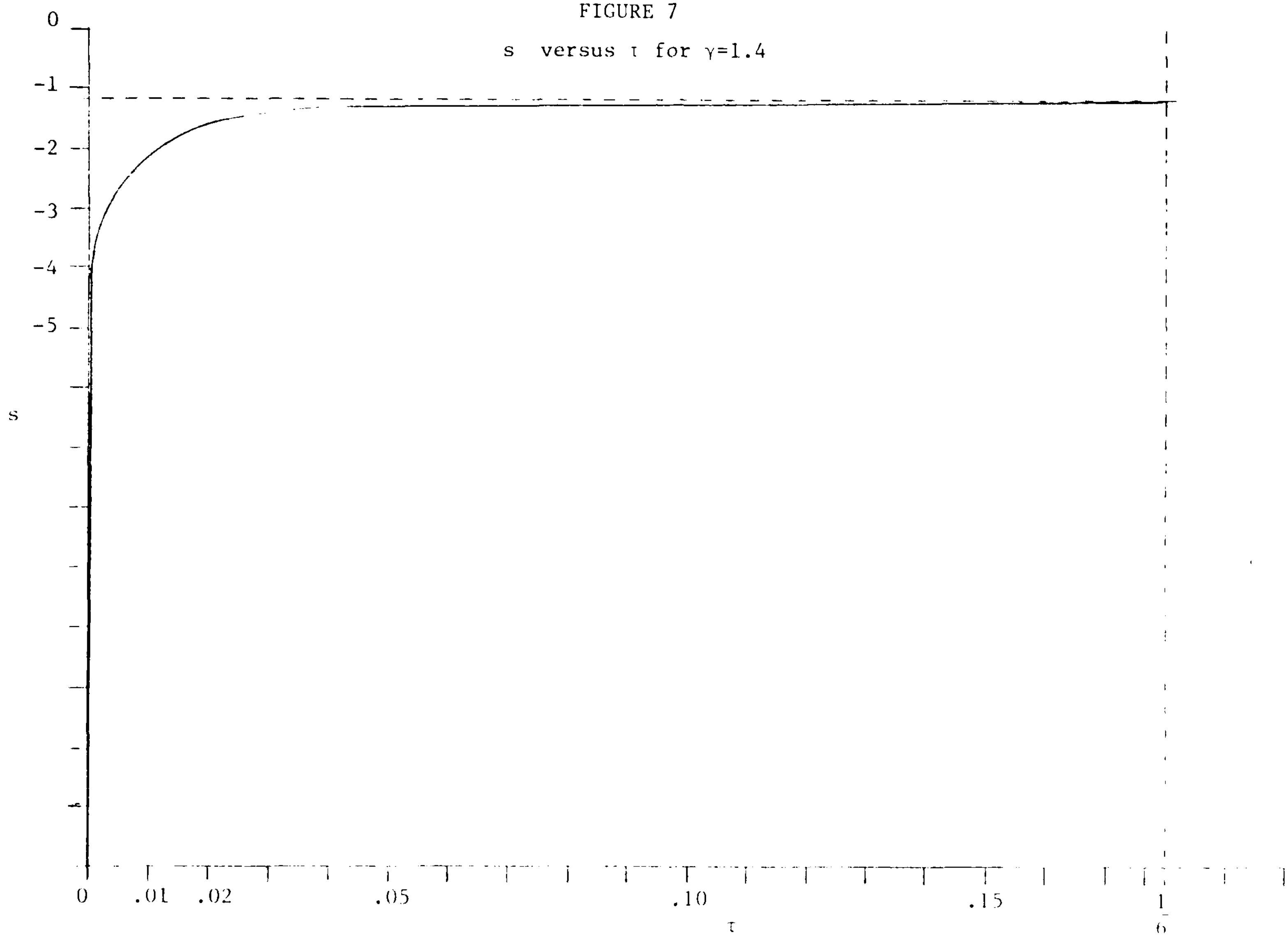
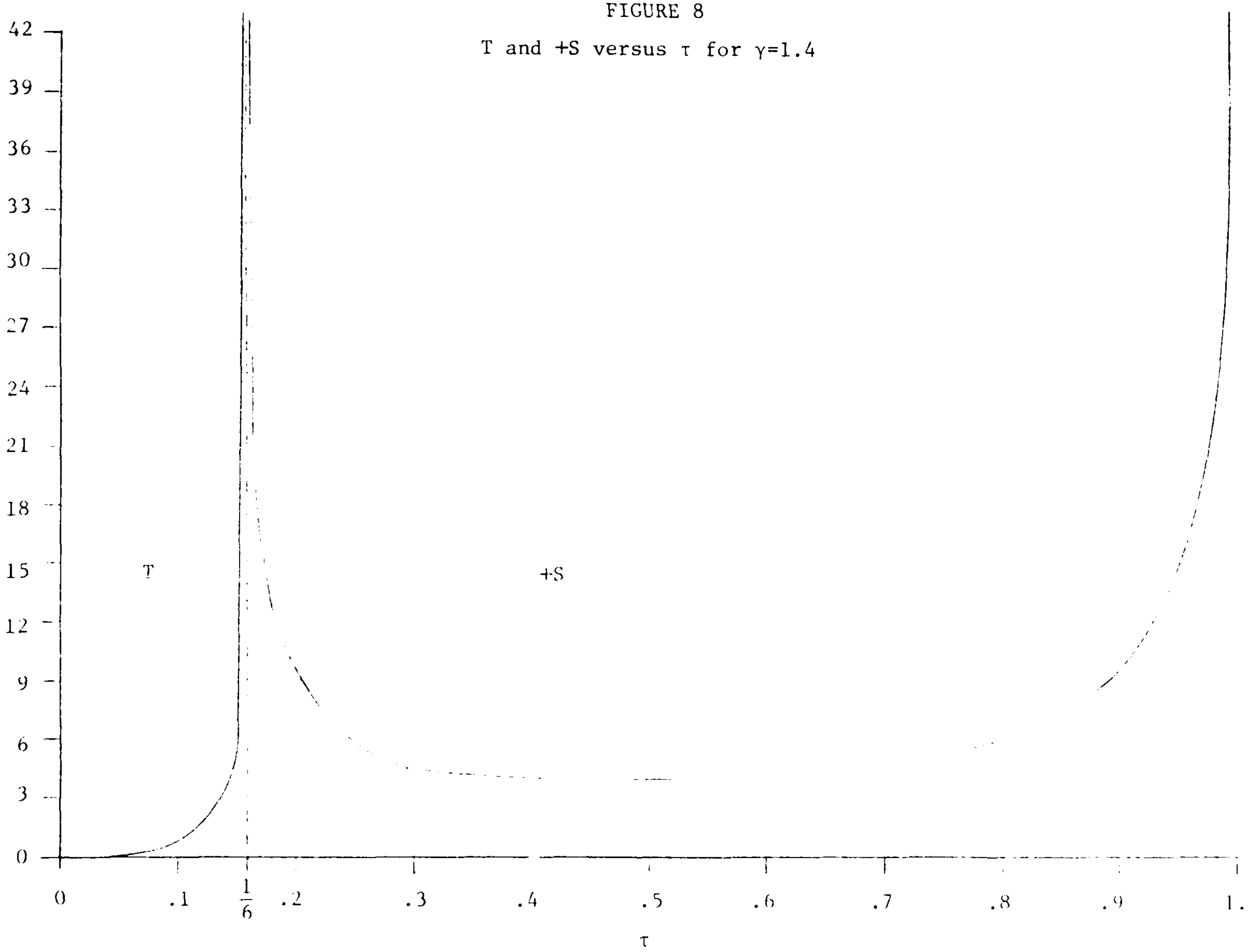


FIGURE 8

T and +S versus τ for γ=1.4



Lighthill[3] develops theorem 2 by assuming a form of $\psi_n(\tau)$ amenable to a series solution of the governing equation of subsonic flow, equation (D.17).

First integrate equation (D.18)

$$1/2\ell n|P/Q| \Big|_{1}^{-P/Q} = \int_{-\infty}^{s} T(s_1)ds_1 \qquad (D.32)$$

$$1/2\ell n|-P/Q| = \int_{-\infty}^{s} T(s_1)ds_1 \qquad (D.33)$$

$$(-P/Q)^{1/2} = EXP\{\int_{-\infty}^{s} T(s_1)ds_1\} \qquad (D.34)$$

Define,

$$V(\tau) = (-P/Q)^{1/4} = EXP\{1/2\int_{-\infty}^{s} T(s_1)ds_1\} \qquad (D.35)$$

where the additional factor of 1/2 is needed in the series solution of equation (D.17). Thus,

$$V(o) = 1 \qquad (D.36)$$

$$dV(\tau)/ds = 1/2T(s)V(\tau) \ . \qquad (D.37)$$

Lighthill[3] assumes

$$\psi_n(\tau) = e^{ns}V(\tau)W_n(s) \ . \qquad (D.38)$$

Note that $e^{ns} = \tau^{n/2}$ and thus $V(\tau)W_n(s)$ must represent $F(a_n, b_n; n+1; \tau)$. Because $\psi_n(\tau)$, e^{ns}, $V(\tau)$ are all analytic functions of τ in the given domain ($\tau \neq o$, τ_s), then $W_n(s)$ must also be an analytic function of τ, and hence of e^{2s}. Also, when $\tau = o$, $F(a_n, b_n; n+1; \tau) = 1$. Because $V(o) = 1$, so too must $W_n(-\infty) = 1$. This implies

$$W_n(s) = \sum_{r=o}^{\infty} d_{n,r}e^{2rs} \qquad (D.39)$$

$$d_{n,o} = 1 \tag{D.40}$$

Subjecting equation (D.38) to the linear operator of equation (D.17) yields:

$$d^2W_n(s)/ds^2 + 2n\,\partial W_n(s)/ds = [1/4T^2(s) - 1/2\,dT(s)/ds]\,W_n(s)\ . \tag{D.41}$$

Were it not for the factor of 1/2 in the definition of V, extra terms of $nT(s)W_n(s)$ and $T(s)dW_n(s)/ds$ appear, rendering the following solution impossible.

Lighthill[3] next states that, by lemma 1,

$$1/4\ T^2(s) - 1/2dT(s)/ds = \sum_{r=2}^{\infty} t_r e^{2rs} \text{ for } |e^{2s}| < e^{2\sigma}\ . \tag{D.42}$$

Equation (D.41) becomes:

$$\sum_{r=1}^{\infty} d_{n,r} \cdot 4r^2 e^{2rs} + 2\sum_{r=1}^{\infty} d_{n,r} \cdot 2re^{2rs} = \sum_{r=2}^{\infty} t_r e^{2rs} \sum_{r=o}^{\infty} d_{n,r} e^{2rs}\ . \tag{D.43}$$

Matching powers of e shows that $d_{n,1} = o$ and

$$4r(n+r)d_{n,r} = \sum_{m=o}^{r-2} t_{r-m} d_{n,m}\ ;\ r \geq 2\ . \tag{D.44}$$

The remainder of this appendix is strictly from Lighthill[3]. Now let $\sigma_1, \sigma_2, \sigma_3$ be any numbers satisfying $\sigma_1 < \sigma_2 < \sigma_3 < \sigma$. Then

$$\sum_{r=2}^{\infty} t_r e^{2r\sigma_3} \tag{D.45}$$

converges, so its terms are bounded and one can write $|t_r| < Ae^{-2r\sigma_3}$.

Lemma 3. If $m > o$, and $|n| > AC/\delta$, where $C = 1/(e^{2(\sigma_3 - \sigma_2)} - 1)$, and n is restricted to lie at a distance $> \delta$ from any negative integer (where $o < \delta < 1$), then

$$|d_{n,m}| < (A/\delta|n|)e^{-2m\sigma_2}\ . \tag{D.46}$$

Proof: Assume it true for m = 1, 2, 3, . . . , r-1. Then by (D.46)

$$4r(n+r) \cdot Ae^{-2r\sigma_3} + \sum_{m=1}^{r-2} \frac{A}{n} e^{-2m\sigma_2} Ae^{-2(r-m)\sigma_3} \qquad (D.47)$$

$$= Ae^{-2r\sigma_3} = \frac{A^2 e^{-2r\sigma_3}}{n} \sum_{m=1}^{r-2} e^{2m(\sigma_3-\sigma_2)} \qquad (D.48)$$

$$4r(n+r) \cdot Ae^{-2r\sigma_3} + \frac{A^2}{n} e^{-2r\sigma_3} Ce^{2r(\sigma_3-\sigma_2)} \qquad (D.49)$$

Now $4r(n+r)| > 2|n|\delta$, for when $|r| > |n/2|$ it is $> 4|n/2|\delta$, and when $|r| < |n/2|$ it is $> 4|n/2|$. Hence,

$$|d_{n,r}| < Ae^{-2r\sigma_2}\left(1+\frac{AC}{|n|}\right)\frac{1}{2|n|\delta} < \frac{Ae^{-2r\sigma_2}}{|n|} \qquad (D.50)$$

if $|n| > AC/\delta$. Hence the result is true and the lemma follows.

It is deduced that, for $s \leq \sigma_1$, under the restrictions of lemma 3,

$$|W_n(s)-1| \leq \sum_{r=1}^{\infty} |d_{n,r}| e^{2r\sigma_1} \qquad (D.51)$$

$$< A/\delta|n| \;.\; 1/1-e^{2(\sigma_1-\sigma_2)} \qquad (D.52)$$

and thus $W_n(s) \longrightarrow 1$ and theorem 2 results. One would do well to note that the sum after N terms, S_N , of the series

$$\sum_{n=o}^{\infty} ar^n$$

is: $S_N = a(1-r^N)/(1-r)$

when following Lighthill's above procedure. Note also the fortuitous form of $4r(n+r)$. For $n = -r$ the above is invalid because $|d_{n,r}|$ is indeterminate. However, $\psi_n(\tau)$ has simple poles at the negative real integers (except $n = -1$) and Lighthill is entirely valid in excluding circles of radius δ about each negative real integer in this analysis.

END